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On the number of non-dominated points of a multicriteria optimization problem☆

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ABSTRACT

This work proposes an upper bound on the maximal number of non-dominated points of a multicriteria optimization problem. Assuming that the number of values taken on each criterion is known, the criterion space corresponds to a comparability graph or a product of chains. Thus, the upper bound can be interpreted as the stability number of a comparability graph or, equivalently, as the width of a product of chains. Standard approaches or formulas for computing these numbers are impractical. We develop a practical formula which only depends on the number of criteria. We also investigate the tightness of this upper bound and the reduction of this bound when feasible, possibly efficient, solutions are known.

1. Introduction

In multicriteria optimization, in opposition to single criterion optimization, there is typically no optimal solution i.e., one that is best for all the criteria. Therefore, the standard situation is that any solution can always be improved on at least one criterion. The solutions of interest, called *efficient* solutions, are those such that any other solution which is better on one criterion is necessarily worse on at least one other criterion. In other words, a solution is efficient if its corresponding vector of criterion values is not dominated by any other vector of criterion values corresponding to a feasible solution. These vectors, associated to efficient solutions, are called *non-dominated points*. For many multicriteria optimization problems, one of the main difficulties is the large cardinality of the set of non-dominated points, and the even larger cardinality of the set of efficient solutions (considering that several solutions can have the same image in the criterion space). However, similarly to single criterion optimization where we usually look for one among all optimal solutions, we usually look for all non-dominated points and a corresponding efficient solution for each such point. Thus, we can restrict our study to the set of non-dominated points. Even with this restriction, it is well-known, that most multicriteria combinatorial optimization problems are *intractable*, in the sense that they admit families of instances for which the number of non-dominated points is exponential in the size of the instance [4]. This situation arises when the number of values taken on each criterion is itself exponential in the size of the instance. It is thus interesting to investigate the number of non-dominated points when we

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know (or have an upper bound on) the number of values taken on each criterion. This problem can be stated within different theoretical frameworks. Using graph theory, the maximal cardinality of a set of non-dominated points corresponds to the *stability number* of a given graph. Using ordered set theory, this maximal cardinality corresponds to the *width* of a *product of chains*. These two frameworks provide different insights on our problem.

Up to our knowledge, this problem has not been dealt with, except very recently by Stanojević et al. in [9]. The best bound they give is obtained by a recursion formula which is well-known in ordered set theory [8] and that we recall in our Proposition 1. Unfortunately, this formula becomes quickly impractical when the number of values on each criterion increases. One of our purposes is to provide an alternative formula which does not depend on these numbers.

In the following section, we define the basic concepts and formalize the problem both in the context of graphs and ordered sets. Then, in Section 3, we deal with simple cases and provide, in the general case, a formula using a combinatorial version of the inclusion–exclusion principle [2]. The time for computing this formula is only exponential in the number of criteria. We also make comparisons with other bounds which are easier to compute. In Section 4, we show that the proposed bound is tight for many classical multicriteria optimization problems. In Section 5, we try to reduce the maximal number of non-dominated points using known feasible solutions, possibly efficient. We conclude with some possible extensions to this work.

2. Basic concepts and problem statements

2.1. Basic concepts

In this paper, we consider multicriteria optimization problems formulated as:

$$\min_{x\in S}\{f_1(x), \ldots, f_p(x)\}, \tag{1}$$

where $f_1, \ldots, f_p$ are $p \geq 2$ criterion functions to be minimized and S is the set of feasible solutions.

We distinguish the decision space X which contains the set S of feasible solutions from the criterion space $Y \subseteq R^p$ which contains the criterion vectors associated to these solutions. We denote by $f(x) = (f_1(x), \ldots, f_p(x))$ the feasible point associated to a feasible solution $x \in S$, and by $Z = f(S)$ the set of images of the feasible solutions. We define in the criterion space Y, the following partial strict order, denoted by $\leq$, such that for any $y, y' \in Y$, $y \leq y'$ if $y_i \leq y'_i$ for all $i \in \{1, \ldots, p\}$ and $y \neq y'$. Relation $\leq$ corresponds to the standard dominance relation used in multicriteria optimization.

Then we define *efficient* solutions and *non-dominated* points, respectively, in the decision space X and in the criterion space Y, as follows:

Definition 1. A feasible solution $x \in S$ is called *efficient* if there is no other feasible solution $x' \in S$ such that $f(x') \leq f(x)$. We denote by S_{Eff} the set of efficient solutions. If x is efficient, $f(x)$ is a *non-dominated point* in the criterion space, and let $Z_{ND} = f(S_{Eff})$.

In this context formulation (1) means that we aim at generating the set of all non-dominated points and a corresponding efficient solution for each such point.

In this paper, we assume that f_i can take up to $c_i + 1$ values, where c_i is a nonnegative integer. Thus, we consider, without loss of generality, that each f_i can take integer values between 0 and c_i, $i = 1, \ldots, p$.

In some cases, the c_i values are known precisely, e.g., for qualitative criteria which take values on a scale whose grades correspond to predefined judgments. In other cases, these values can only be approximated. For instance, assuming that criterion functions are integer-valued, we can find an upper bound on c_i by computing the coordinates of the *ideal* and *anti-ideal* points, corresponding, respectively, to the best and the worst possible values on each criterion. Better bounds can be given if we can compute the coordinates of the *nadir* point, which corresponds to the worst possible values over the set of non-dominated points. Unfortunately, this is not easy in general, especially when the number of criteria is at least 3 [5].

The problem of determining the maximum cardinality of the non-dominated set can be stated as follows.

MAX SIZEND

Input: an integer p and p integers c_i, $i = 1, \ldots, p$.

Output: maximum cardinality of the non-dominated set Z_{ND} associated to a set Z of p-dimensional points such that at most $c_i + 1$ values are taken on the ith dimension, $i = 1, \ldots, p$.

Let $(\overline{c_i + 1}) = \{0, \ldots, c_i\}$, $i = 1, \ldots, p$ and $P = (\overline{c_1 + 1}) \times \cdots \times (\overline{c_p + 1})$. Any relevant set Z, and in particular any of those leading to a non-dominated set of maximum cardinality, is included in P.

2.2. Statement as a graph theory problem

Consider the graph $G = (P, E)$ whose set of vertices is $P = (\overline{c_1 + 1}) \times \cdots \times (\overline{c_p + 1})$ and set of edges is $E = \{(u, v) \in P \times P : u \leq v\}$. By construction, G is a *comparability* graph (i.e., a graph that admits a transitive orientation), since relation $\leq$ is transitive.