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[1, 2]-sets in graphs

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ABSTRACT

A subset $S \subseteq V$ in a graph $G = (V, E)$ is a $[j, k]$-set if, for every vertex $v \in V \setminus S$, $j \leq |N(v) \cap S| \leq k$ for non-negative integers j and k, that is, every vertex $v \in V \setminus S$ is adjacent to at least j but not more than k vertices in S. In this paper, we focus on small j and k, and relate the concept of $[j, k]$-sets to a host of other concepts in domination theory, including perfect domination, efficient domination, nearly perfect sets, 2-packings, and k-dependent sets. We also determine bounds on the cardinality of minimum [1, 2]-sets, and investigate extremal graphs achieving these bounds. This study has implications for restrained domination as well. Using a result for [1, 3]-sets, we show that, for any grid graph G, the restrained domination number is equal to the domination number of G.

1. Introduction

Let $G = (V, E)$ be a graph of *order* $n = |V|$ and *size* $m = |E|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$ of vertices adjacent to v. Each vertex in $N(v)$ is called a *neighbor* of v. The *degree* of a vertex v is $\deg(v) = |N(v)|$. A vertex of degree 1 is called a *leaf*, and its neighbor is called a *support vertex*. The minimum and maximum degrees of a vertex in a graph G are denoted $\delta(G)$ and $\Delta(G)$, respectively. We say that a graph G is *k-regular* if every vertex in G has $\deg(v) = k$. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$. Let $S \subseteq V$ and $v \in S$. The *S-private neighborhood* of v, denoted by $\mathrm{pn}_G[v, S]$ or simply by $\mathrm{pn}[v, S]$ if the graph G is clear from the context, consists of all vertices in the closed neighborhood of v but not in the closed neighborhood of $S \setminus \{v\}$; that is, $\mathrm{pn}[v, S] = N[v] \setminus N[S \setminus \{v\}]$. Thus, if $u \in \mathrm{pn}[v, S]$, then $N[u] \cap S = \{v\}$.

A set S is a *dominating set* of a graph G if $N[S] = V$, that is, for every $v \in V$, either $v \in S$ or $v \in N(u)$ for some vertex $u \in S$. The minimum cardinality of a dominating set in a graph G is called the *domination number*, and is denoted $\gamma(G)$. Given two disjoint sets of vertices $R, S \subset V$, with $R \cap S = \emptyset$, we say that R *dominates* S if $S \subseteq N(R)$, or, equivalently, if every vertex in S is adjacent to at least one vertex in R. A dominating set that is independent is an independent dominating set, and the minimum cardinality of an independent dominating set of G is the *independent domination number* $i(G)$. A dominating set with minimum cardinality is called a $\gamma(G)$-set, and an independent dominating set with minimum cardinality is called a $i(G)$-set.

We define a subset $S \subseteq V$ in a graph $G = (V, E)$ to be a $[j, k]$*-set* if, for every vertex $v \in V \setminus S$, $j \leq |N(v) \cap S| \leq k$, that is, every vertex in $V \setminus S$ is adjacent to at least j vertices, but not more than k vertices in S. For $j \geq 1$, a $[j, k]$-set S is a

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Table 1
Types of [0, 1]-sets and [1, 1]-sets.

Which vertices	Bounds	Type of set
$\forall v \in V \setminus S$	$0 \le \lvert N(v) \cap S\rvert \le 1$	Nearly perfect
$\forall v \in V$	$0 \le \lvert N[v] \cap S\rvert \le 1$	2-packing
$\forall v \in V$	$0 \le \lvert N(v) \cap S\rvert \le 1$	Open 2-packing
$\forall v \in V \setminus S$	$1 \le \lvert N(v) \cap S\rvert \le 1$	Perfect dominating; 1-fair dominating
$\forall v \in V$	$1 \le \lvert N[v] \cap S\rvert \le 1$	Efficient dominating
$\forall v \in V$	$1 \le \lvert N(v) \cap S\rvert \le 1$	Open efficient dominating

Table 2
Types of [1, 2]-sets and [0, 2]-sets.

Which vertices	Bounds	Type of set
$\forall v \in V \setminus S$	$1 \le \lvert N(v) \cap S\rvert \le 2$	[1, 2]-set
S independent and $\forall v \in V \setminus S$	$1 \le \lvert N(v) \cap S\rvert \le 2$	Independent [1, 2]-set
$\forall v \in V$	$1 \le \lvert N[v] \cap S\rvert \le 2$	1-dependent [1, 2]-set
$\forall v \in V$	$1 \le \lvert N(v) \cap S\rvert \le 2$	Total [1, 2]-set
$\forall v \in V \setminus S$	$0 \le \lvert N(v) \cap S\rvert \le 2$	Nearly 2-perfect
S independent and $\forall v \in V \setminus S$	$0 \le \lvert N[v] \cap S\rvert \le 2$	Independent nearly 2-perfect
$\forall v \in V$	$0 \le \lvert N[v] \cap S\rvert \le 2$	1-dependent nearly 2-perfect
$\forall v \in V$	$0 \le \lvert N(v) \cap S\rvert \le 2$	2-dependent nearly 2-perfect

dominating set, since every vertex in $V \setminus S$ has at least one neighbor in S (is dominated by S). The study of $[j, k]$-sets, for $j \ge 1$, is motivated by the need to have dominating sets, for example, acting as servers in a computing network, or acting as sets of monitoring devices in situations requiring surveillance, but with the need to establish such sets as efficiently or as cost effectively as possible, that is, without creating too much redundancy. A generalization of this concept, called *set-restricted domination*, was introduced by Amin and Slater in [2,3]. For set-restricted domination, let $V(G) = \{v_1, v_2, \ldots, v_n\}$, and assign to each v_i a set S_i of non-negative integers, which specifies that the number of times that vertex v_i is to be dominated must be an integer in S_i. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$. If there exists a set $S \subseteq V$ such that $|N[v_i] \cap S| \in S_i$ for $1 \le i \le n$, then S is a *set-restricted dominating set*, and $\gamma_S(G)$ is the *set-restricted domination number*. Let S be a set-restricted dominating set. Then, for example, we have the following.

1. If every $S_i = \{1, 2, \ldots, n\}$, then $\gamma_S(G) = \gamma(G)$.
2. If $S_i = \{k, k+1, \ldots, n\}$ for $1 \le i \le n$, then S is a *k-dominating set*. Multiple domination was introduced by Fink and Jacobson in [24].
3. If $S_i = \{1, 3, 5, \ldots\}$ for $1 \le i \le n$, or if $S_i = \{2, 4, 6, \ldots\}$ for $1 \le i \le n$, then S is a *parity-restricted dominating set*. Parity-restricted dominating sets were the focus of [2,3].
4. If $S_i = \{1, 2\}$ for $1 \le i \le n$, then S is a *quasi-perfect dominating set*, as introduced by Dejter [16].

Thus, a $[1, k]$-set is a variant of a set-restricted dominating set in which (i) you only require that for every vertex $v_i \in V \setminus S$, $|N[v_i] \cap S| \in S_i$, and (ii) you require that $S_i = \{1, 2, \ldots, k\}$ for all $1 \le i \le n$. Thus, $[j, k]$-domination, in particular, $[1, k]$-domination, has not been investigated. A different idea using similar notation was introduced in 1988 by Sampathkumar [32], where he defined a set S to be a $(1, k)$-*dominating set* if S is a dominating set and every vertex in S dominated at most k vertices in $V \setminus S$, that is, $|N(v) \cap (V \setminus S)| \le k$ for all $v \in S$.

Definition 1. For $k \ge 1$, the $[1, k]$-*domination number* in a graph G, denoted $\gamma_{[1,k]}(G)$, equals the minimum cardinality of a $[1, k]$-set in G. A $[1, k]$-set with cardinality $\gamma_{[1,k]}(G)$ is called a $\gamma_{[1,k]}(G)$-*set*.

In this paper, we focus primarily on [1, 2]-sets in graphs, which can be related to a number of other concepts in domination theory, as follows. We illustrate types of [0, 1]-sets, [1, 1]-sets, [1, 2]-sets, and [0, 2]-sets in the following tables. Definitions of the parameters in the tables can be found in the Appendix.

This study has implications for restrained domination as well. For a graph $G = (V, E)$, set $S \subseteq V$ is *restrained dominating set* if it is a dominating set for which every vertex $v \in V \setminus S$ has at least one neighbor $w \in V \setminus S$. The *restrained domination number* $\gamma_r(G)$ equals the minimum cardinality of a restrained dominating set in G. This type of domination was introduced by Domke et al. in [21].

The *girth* of a graph G, denoted g, equals the minimum length of an induced cycle in G. The following observations are easily derived from the definitions in Tables 1 and 2.

1. For any graph G, $\gamma(G) \le \gamma_{[1,2]}(G) \le \gamma_p(G)$, where $\gamma_p(G)$ denotes the minimum cardinality of a perfect dominating set in G.
2. If G is a 3-regular graph of order n and girth g, then $\gamma_{[1,2]}(G) \le n - g$.
3. If G is a 3-regular graph, then $\gamma_r(G) = \gamma_{[1,2]}(G)$.
4. If G is a 4-regular graph of order n and girth g, then $\gamma_{[1,2]}(G) \le n - g$.
5. If $\gamma(G) \le 2$, then $\gamma(G) = \gamma_{[1,2]}(G)$.
6. For the complement $\overline{K}_n$ of the complete graph K_n, $\gamma(\overline{K}_n) = \gamma_{[1,2]}(\overline{K}_n) = n$.