Discrete Applied Mathematics 161 (2013) 2910–2924

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

New bounds on the minimum density of an identifying code for the infinite hexagonal grid☆

Ari Cukierman, Gexin Yu*

Department of Mathematics, College of William and Mary, Williamsburg, VA, 23185, United States

ARTICLE INFO

Article history:
Received 18 October 2011
Received in revised form 31 May 2013
Accepted 3 June 2013
Available online 25 June 2013

Keywords:
Identifying code
Infinite hexagonal grid
Discharging method

ABSTRACT

For a graph, G, and a vertex $v \in V(G)$, let $N[v]$ be the set of vertices adjacent to and including v. A set $D \subseteq V(G)$ is a (vertex) identifying code if for any two distinct vertices $v_1, v_2 \in V(G)$, the vertex sets $N[v_1] \cap D$ and $N[v_2] \cap D$ are distinct and non-empty. We consider the minimum density of a vertex identifying code for the infinite hexagonal grid. In 2000, Cohen et al. constructed two codes with a density of $\frac{3}{7} \approx 0.428571$, and this remains the best known upper bound. Until now, the best known lower bound was $\frac{12}{29} \approx 0.413793$ and was proved by Cranston and Yu in 2009. We present three new codes with a density of $\frac{3}{7}$, and we improve the lower bound to $\frac{5}{12} \approx 0.416667$.

1. Introduction

The study of (vertex) identifying codes is motivated by the desire to detect failures efficiently in a multi-processor network. Such a network can be modeled as an undirected graph, G, where $V(G)$ represents the set of processors and $E(G)$ represents the set of connections among processors. Suppose we place detectors on a subset of these processors. These detectors monitor all processors within a neighborhood of radius r and send a signal to a central controller when a failure occurs. We assume that no two failures occur simultaneously. A signal from a detector, d, indicates that a processor in the r-neighborhood of d has failed but provides no further information. Now, any given processor, p, might be in the r-neighborhood of several detectors, $d_1, d_2, d_3 \dots$. Then, when p fails, the central controller receives signals from $d_1, d_2, d_3 \dots$. Let us call $\{d_1, d_2, d_3, \dots\}$ the identifying set of p in G. If each processor has a unique and non-empty trace, then the central controller can determine which processor failed simply by noting the detectors from which signals were received. In this case, we call the subset of processors on which detectors were placed an identifying code.

Vertex identifying codes were first introduced in 1998 by Karpovsky, Chakrabarty and Levitin [7]. The processors of the preceding paragraph become the vertices of a graph, and the processors on which detectors have been placed become the vertex subset called a vertex identifying code. In the example above, we considered detectors which monitor a neighborhood of radius r. In this paper, we concern ourselves with the case in which $r = 1$.

Let $N_i(v)$ be the set of vertices at distance-i from a vertex, v, and let $N[v] = N_1(v) \cup \{v\}$.

Definition 1.1. Consider a graph, G. A set $D \subseteq V(G)$ is an *identifying code* if

(i) for all $v \in V(G)$, $N[v] \cap D \neq \emptyset$
(ii) for all $v_1, v_2 \in V(G)$ where $v_1 \neq v_2$, $N[v_1] \cap D \neq N[v_2] \cap D$.

☆ The research is supported in part by NSF grant CSUMS grant DMS-0703532, NSF grant DMS-0852452, and NSA grant H98230-12-1-0226.
* Corresponding author. Tel.: +1 757 221 2040.
E-mail address: gyu@wm.edu (G. Yu).

0166-218X/$ – see front matter
http://dx.doi.org/10.1016/j.dam.2013.06.002

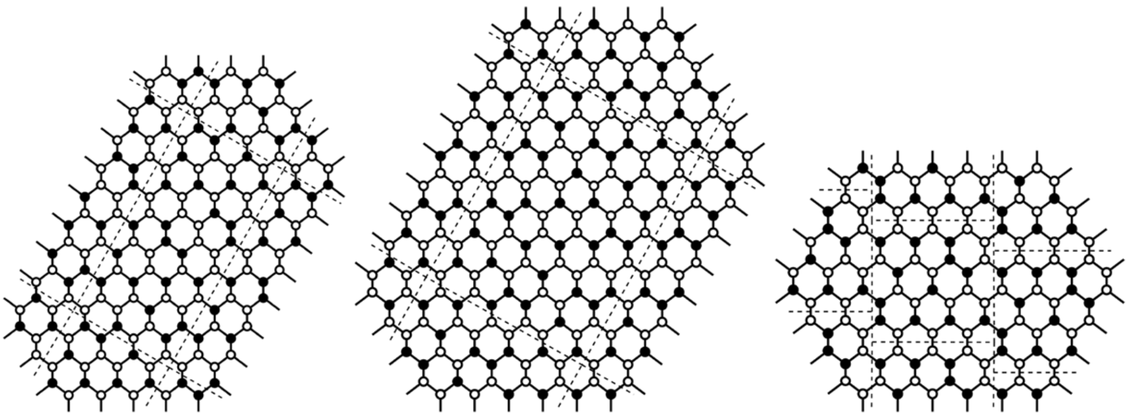

Fig. 1.1. Three new codes with a density of 3/7. The solid vertices are in the code.

From Definition 1.1, we see that some graphs do not admit vertex identifying codes. In particular, if $N[v_1] = N[v_2]$ for some distinct $v_1, v_2 \in V(G)$ then G does not admit a vertex identifying code because $N[v_1] \cap D = N[v_2] \cap D$ for any $D \subseteq V(G)$. On the other hand, if $N[v_1] \neq N[v_2]$ for all distinct $v_1, v_2 \in V(G)$ then G admits a vertex identifying code because $V(G)$ is such a code.

Of particular interest are vertex identifying codes of minimal cardinality. When dealing with infinite graphs, we consider instead the *density* of a vertex identifying code, i.e., the ratio of the number of vertices in the code to the total number of vertices. Let G be an infinite graph, and let $D \subseteq V(G)$ be a vertex identifying code for G. Then, for some $v \in V(G)$, the set of vertices in D within distance-k of v is given by $\bigcup_{i=0}^{k} N_i(v) \cap D$. Let $\sigma(D, G)$ be the density of D in G. Then,

$$\sigma(D, G) = \limsup_{k\to\infty} \frac{\left|\bigcup_{i=0}^{k} N_i(v) \cap D\right|}{\left|\bigcup_{i=0}^{k} N_i(v)\right|}. \tag{1.1}$$

Let $\sigma_0(G)$ be the minimum density of a vertex identifying code for G; that is,

$$\sigma_0(G) = \min_{D}\{\sigma(D, G)\}. \tag{1.2}$$

Karpovsky et al. [7] considered the minimum density of vertex identifying codes for the infinite triangular (G_T), square (G_S) and hexagonal (G_H) grids. They showed $\sigma_0(G_T) = 1/4$. In 1999, Cohen et al. [2] proved $\sigma_0(G_S) \leq 7/20$, and, in 2005, Ben-Haim and Litsyn [1] completed the proof by showing $\sigma_0(G_S) \geq 7/20$.

We concern ourselves in this paper with $\sigma_0(G_H)$. In 1998, Karpovsky et al. [7] showed $\sigma_0(G_H) \geq 2/5 = 0.4$. In 2000, Cohen et al. [3] improved this result to $\sigma_0(G_H) \geq 16/39 \approx 0.410256$ and constructed two codes with a density of $3/7 \approx 0.428571$ implying $\sigma_0(G_H) \leq 3/7$. In 2009, Cranston and Yu [4] proved $\sigma_0(G_H) \geq 12/29 \approx 0.413793$. It should be noted that the optimal results have been obtained for r-identifying codes with $r \geq 2$ for the hexagonal grid, see [5,6,8,9].

In this paper, we present three new codes with a density of 3/7 and prove $\sigma_0(G_H) \geq 5/12 \approx 0.416667$. In conclusion, it is now known that $5/12 \leq \sigma_0(G_H) \leq 3/7$.

Suppose β is an upper bound on $\sigma_0(G_H)$. To prove this, we need only show the existence of a code, D, with $\sigma(D, G_H) \leq \beta$. When constructing such codes, we usually look for tiling patterns. Since the pattern repeats ad infinitum, the density of one tile is the density of the whole graph. Fig. 1.1 shows three new codes for the infinite hexagonal grid with a density of 3/7.

Theorem 1.2. *The minimum density of a vertex identifying code for the infinite hexagonal grid is greater than or equal to* 5/12.

To prove Theorem 1.2, we employ the discharging method. Let D be an arbitrary vertex identifying code for G_H. We assign 1 "charge" to each vertex in D which we then redistribute so that every vertex in G_H retains at least 5/12 charge. The charge is redistributed in accordance with a set of "Discharging Rules". Since D was chosen arbitrarily, we then conclude that 5/12 is a lower bound on $\sigma_0(G_H)$.

As the proof of Theorem 1.2 is rather lengthy, we include a sketch of the proof in Section 2. In Section 3, we introduce several properties of vertex identifying codes for G_H which we will reference throughout the paper. Section 4 is devoted to terminology and notations; the vast majority of relevant notions are defined here. In Section 5, we state several lemmas concerning the structure of vertex identifying codes for G_H. However, we omit the proofs of these lemmas (interested readers can find them at http://arxiv.org/abs/1110.1097). The main result of this paper, Theorem 1.2, is proved in Section 6.

As an additional remark, we do not think a more complicated analysis using discharging method will produce results better than 5/12. A different approach should be used to improve the lower bound, or a construction should be found with density 5/12.