Discrete Applied Mathematics 161 (2013) 3063–3071

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The Laplacian polynomial and Kirchhoff index of graphs derived from regular graphs☆

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ARTICLE INFO

Article history:
Received 15 June 2012
Received in revised form 3 June 2013
Accepted 6 June 2013
Available online 29 June 2013

Keywords:
Resistance distance
Kirchhoff index
Laplacian spectrum
Bound

ABSTRACT

Let $R(G)$ be the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the corresponding edge, and $Q(G)$ be the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. In this paper, we determine the Laplacian polynomials of $R(G)$ and $Q(G)$ of a regular graph G; on the other hand, we derive formulae and lower bounds of the Kirchhoff index of these graphs.

1. Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote by $A(G)$ and $D(G)$ the adjacency matrix and the diagonal matrix with the vertex degrees of G on the diagonal, respectively. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G, for details see [22,23]. Denote by $P_G(\lambda)$ and $\mu_G(\lambda)$ the adjacent characteristic polynomial $\det(\lambda I_n - A(G))$ and the Laplacian characteristic polynomial $\det(\lambda I_n - L(G))$ of G, respectively. The multiset of eigenvalues of $A(G)$ (resp., $L(G)$) are called the adjacency (resp., Laplacian) spectrum of G. Since $A(G)$ and $L(G)$ are all real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ (resp., $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$) are the adjacency (resp., Laplacian) eigenvalues of G. Clearly, all Laplacian eigenvalues of G are non-negative. If the graph G is connected, then $\mu_i(G) > 0$ for $i = 1, 2, \ldots, n-1$ and $\mu_n(G) = 0$ [13,14,22]. In what follows, the Laplacian spectrum of G is denoted by $S(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$.

Suppose G is a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Define two graph operators R and Q (see the definitions in p. 63 in [5]) as follows. Let $R(G) = (V(R(G)), E(R(G)))$ be the graph obtained from G by adding a new vertex e' corresponding to each edge $e = (a, b)$ of G and by joining each new vertex e' to the end vertices a and b of the corresponding edge $e = (a, b)$, i.e., $R(G)$ is obtained from G by "changing each edge $e = (a, b)$ of G into a triangle $ae'b$". Thus, $V(R(G)) = V(G) \cup \{e' \mid e \in E(G)\}$ and $E(R(G)) = E(G) \cup \{(v_i, e'), (v_j, e') \mid e = (v_i, v_j) \in E(G)\}$ (see Fig. 1(a) and (b) for example). Let $Q(G) = (V(Q(G)), E(Q(G)))$ be the graph obtained from G by inserting a new vertex e_i' into every edge e_i of G and by joining by edges those pairs of these new vertices e_i' and e_j' which lie on adjacent edges e_i and e_j of G, $i, j = 1, 2, \ldots, m$. Denote by v_{i1} and v_{i2} the end-vertices of edge e_i of G. Then $V(Q(G)) = V(G) \cup \{e_i' \mid e_i \in E(G),$

☆ This research was partially supported by the National Natural Science Foundation of China (No. 10971086 and No. 11201201).

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http://dx.doi.org/10.1016/j.dam.2013.06.010

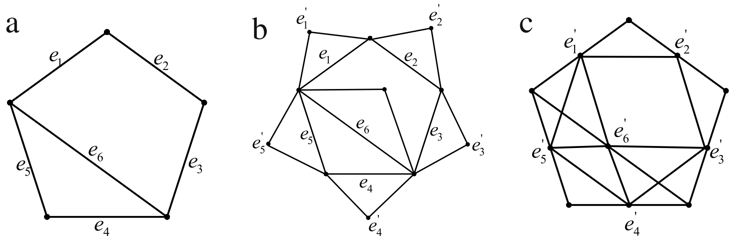

Fig. 1. (a) The graph G. (b) The graph $R(G)$. (c) The graph $Q(G)$.

$i = 1, 2, \ldots, m\}$ and $E(Q(G)) = \{(v_{i1}, e_i'), (v_{i2}, e_i') | i = 1, 2, \ldots, m\} \cup \{(e_i', e_j') \mid e_i$ and e_j are adjacent edges of $G\}$ (see Fig. 1(a) and (c) for example).

In some graph theory problems it is necessary to compute the spectrum (resp., Laplacian spectrum) of a compound graph, obtained from some operations from some simple graphs. In [5] there exist many relations connecting the spectrum (resp., Laplacian spectrum) of a compound graph with spectra (resp., Laplacian spectra [17,24]) of graphs from which that graph is derived. However, it is worth considering the corresponding problems of graphs derived from a single graph, such as line graph, subdivision graph, total graph, $R(G)$, $Q(G)$, etc. In [17], the Laplacian polynomial of line graph, the subdivision graph and total graph of the regular graph are obtained. In the present work, on one hand, we determine the Laplacian polynomials of $R(G)$ and $Q(G)$; on the other hand, for these graphs, we compute the topological indices based on the concept of resistance distance.

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The (ordinary) distance between vertices v_i and v_j, denoted by d_{ij}, is the length of a shortest path connecting them. The original index based on distance in a graph G is the Wiener index $W(G)$ [28], which counts the sum of distances between pairs of vertices in G. In 1993, Klein and Randić [20] defined a new distance function named resistance distance framed in terms of electrical network theory. However, this concept has been discussed much earlier (1949) for another purpose by Foster [10] as recently pointed out by Palacios [27].

The resistance distance between vertices v_i and v_j of G, denoted by r_{ij}, is defined to be the effective resistance between nodes v_i and v_j as computed with Ohm's law when all the edges of G are considered to be unit resistors. As an analogue to the Wiener index, the sum $Kf(G) = \sum_{i<j} r_{ij}$ was proposed in [20], later called the Kirchhoff index of G in [4]. Klein and Randić [20] proved that $r_{ij} \le d_{ij}$ with equality if and only if there is exactly one path between v_i and v_j, and so $Kf(G) \le W(G)$ with equality if and only if G is a tree.

Like the Wiener index, the Kirchhoff index has been found noteworthy applications in chemistry, as a molecular structure descriptor [4,6,9,29,34]. However, it is rather hard to implement some algorithms [1,3,20,25,35] to compute resistance distances and Kirchhoff index of a graph. Hence it makes sense to determine bounds or find formulae for the Kirchhoff index for some classes of graphs. In [31], sharp bounds for Kirchhoff index of unicyclic graphs are obtained. H. Zhang et al. [32] characterized the graphs with extremal Kirchhoff index among all n-vertex bicyclic graphs. The Kirchhoff index has also been computed for some classes of graphs, such as cycles [19,21], complete graphs [21], geodetic graphs [26], some fullerenes including buckminsterfullerene [1,2,11], distance transitive graphs [26], and so on [1,18,26,25,27]. The Kirchhoff index of certain composite operations between two graphs was studied as well, such as product, lexicographic product [30] and join, corona, cluster [33]. Another interesting result is the comparison between Kirchhoff index and the Laplacian energy-like invariant [7].

It is of interest to study the Kirchhoff index of graphs derived from a single graph. In [12], the authors obtained formulae and lower bounds of the Kirchhoff index of the line graph, subdivision graph, total graph of a connected regular graph, respectively. The main aim of this paper is to report formulae and lower bounds for the Kirchhoff index of $R(G)$ and $Q(G)$ of regular graph G, respectively. In particular, special formulae are given for the Kirchhoff index of $R(G)$ and $Q(G)$, where G is a complete graph K_n, a cycle C_n and a regular complete bipartite graph $K_{n,n}$.

2. The Laplacian polynomials of $R(G)$ and $Q(G)$

Let G be a regular graph. In this section, we show that the Laplacian polynomial of $R(G)$ and $Q(G)$ are determined by the characteristic polynomial or the Laplacian polynomial of G, respectively. We first list some known results which will be used later.

Lemma 2.1 ([16]). *Let M be a non-singular square matrix. Then*

$$\det \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det M \det(Q - PM^{-1}N).$$

The line graph of a graph G, denoted by $\mathcal{L}(G)$, is the graph whose vertices correspond to the edges of G, with two vertices of $\mathcal{L}(G)$ being adjacent if and only if the corresponding edges of G share a common vertex. The following result is well known [5].