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A tight axiomatization of the median procedure on median graphs

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ABSTRACT

A profile $\pi = (x_1, \ldots, x_k)$, of length k, in a finite connected graph G is a sequence of vertices of G, with repetitions allowed. A median x of π is a vertex for which the sum of the distances from x to the vertices in the profile is minimum. The median function finds the set of all medians of a profile. Medians are important in location theory and consensus theory. A median graph is a graph for which every profile of length 3 has a unique median. Median graphs have been well studied, possess a beautiful structure and arise in many arenas, including ternary algebras, ordered sets and discrete distributed lattices. They have found many applications, for instance in location theory, consensus theory and mathematical biology. Trees and hypercubes are key examples of median graphs.

We establish a succinct axiomatic characterization of the median procedure on median graphs, settling a question posed implicitly by McMorris, Mulder and Roberts in 1998 [19]. We show that the median procedure can be characterized on the class of all median graphs with only three simple and intuitively appealing axioms, namely anonymity, betweenness and consistency. Our axiomatization is tight in the sense that each of these three axioms is necessary. We also extend a key result of the same paper, characterizing the median function for profiles of even length on median graphs.

1. Introduction

Facility location problems involve a set of 'clients' at various locations. One seeks a set of locations acceptable for the provision of a given service. Graphs are a natural model for the locations and interconnections. Hundreds of papers have been written about location problems on graphs using the geodesic metric, see for example the reference lists in [9,21,31]. Let $G = (V, E)$ be a graph. Each client is represented by its preferred location in the graph, so by a vertex. Thus the set of clients may be represented by a sequence, or *profile* $\pi = (x_1, x_2, \ldots, x_k)$. Note that π being a sequence, repetitions of vertices are allowed, by which clients having the same preferred location can be represented. Let V^* be the set of all finite sequences of vertices. The location problem is then modelled by a *location function* $L : V^* \to 2^V - \emptyset$, where 2^V is the power set of V. An appropriate objective function depends on the specific application. To locate a site for an emergency service, one might seek to minimize the greatest distance to any client: hence the center is a good choice. For a facility designed for the delivery of goods, one might want to minimize the average distant to the clients. Here the median set is appropriate. Many versions of 'central' subgraphs have been considered on various classes of graphs, see [13,38,39,33,34,32].

In consensus theory, a finite set, or profile, of voters (users, clients) provides a list of preferences for the outcomes of a decision procedure. One seeks 'consensus', namely a set of outcomes which best satisfy the voters. See the list of references in [37,2,3] for surveys of such social choice functions. The theory of consensus is widely used in e.g. social choice theory, voting theory, economic theory and biomathematics.

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In both settings, that of consensus and that of location, numerous researchers have addressed the issue of identifying an objective function via a succinct 'wish list' of desired properties. The goal here is to identify functions for which this list, or something close, gives a characterization. This method allows one to argue in favor of a particular set of locations (or particular consensus) as being precisely that satisfying certain desirable properties. Another perspective is that one requires that consensus be achieved in a rational way, that is, the objective function should satisfy certain rational rules or 'consensus axioms'. In 1951 Arrow [1] initiated this axiomatic approach for consensus functions by showing that certain sets of axioms could not be satisfied. For a recent survey of this axiomatic approach with an extensive list of references see [7].

Three location functions have been studied axiomatically: the center function, the median function and the mean function. In the literature the terms median function and median procedure have been used interchangeably, quite often both in the same paper. We follow this custom here. For the center function [20,28] and the mean function [11,36,16,17], characterizations have been obtained only on trees and tree networks. In a tree network edges are to be considered as continuous line segments so that interior points of edges are possible locations as well. Characterizations beyond trees seem to be very difficult for these functions. The median and the mean are special instances of the ℓ_p-function, viz. for $p = 1$ and $p = 2$, respectively. Here $\|\pi\|_p = \sqrt[p]{\sum_{i=1}^{k}[d(x, x_i)]^p}$ is minimized, where $\pi = (x_1, x_2, \ldots, x_k)$. This function was studied in [17], and an axiomatization was provided, but again, only for trees.

The median function has been more promising. The first characterization was on tree networks by Vohra [36]. This function satisfies three simple and basic axioms, viz. (A) Anonymity: the clients are anonymous, (B) Betweenness: any location strictly between two clients minimizes the sum of the distances to these two clients, and (C) Consistency: if two sets of clients both prefer location x, then the union of all these clients also prefers location x. It is an easy and well-known result that (A), (B) and (C) are satisfied by the median function for all graphs (in fact for all metric spaces). On most graphs these axioms are not sufficient to characterize the median function. Hence the question arises: *On which graphs is the median function characterized by these three basic axioms*?

In 1998, McMorris et al. [18] showed that the median function on 'cube-free median graphs' is characterized by the three basic axioms. Here cube-free means that the 3-dimensional cube Q_3 does not occur as an induced subgraph. These authors did not show that (A), (B) and (C) suffice for the general class of median graphs: the obstacle in [18] to such an extension was formed by the profiles of even length. However, by proving some nice and surprising results on median sets of even profiles in median graphs, these authors [18] were able to establish that the addition of a fourth 'convexity' axiom, dealing with even profiles, sufficed to yield a characterization of the median function for the class of all median graphs. Unfortunately, this additional axiom is rather complicated and not intuitively appealing. More importantly, the issue of tightness was not settled in their paper: it was left as an open question whether convexity was truly necessary or whether (A), (B) and (C) might in fact suffice for all median graphs. In another paper [14], the Convexity axiom was replaced by another 'heavy duty' axiom: the '$\frac{1}{2}$-condorcet' axiom. But again, it remained an open question whether this fourth axiom was necessary. The main result of the present paper settles this open question by showing that, surprisingly, the three simple and necessary axioms (A), (B) and (C) suffice to characterize the median function for the entire class of all median graphs. As said, this characterization is tight: the three basic axioms are independent. In [27] two simple examples can be found to show that consistency, as well as betweenness, is independent from the two other axioms. A more sophisticated example is necessary to show that anonymity is independent from the other two axioms, see [15]. This example will be included in a future paper.

To appreciate our main result, a word needs to be said about the significance and structure of median graphs.

A median graph is a graph in which any profile of three vertices has a unique median vertex. Median graphs were independently introduced by Avann [4], who called them 'unique ternary distance graphs', by Nebeský [30] and by Mulder and Schrijver [29]. Median graphs are now well studied: see [23,14,26,10] for survey articles. They are important because of the role they play in ternary algebras, ordered sets, discrete lattices, Helly hypergraphs, product graphs and so forth. They have been used in applications in such diverse fields as dynamic search, location theory, social choice theory, biomathematics, mathematical chemistry, computer science, mathematical economics and literary history. Classical examples of median graphs are trees, hypercubes, and grid graphs.

There is a rich and beautiful structure theory for median graphs. Notably, in 1978, Mulder [22] showed that every median graph can be obtained from K_1 by a series of 'convex expansions'. Because we make extensive use of the ideas underlying this operation, we will describe it in detail in the sequel. Trees and hypercubes arise as extreme cases of this expansion procedure and thus median graphs are a very natural common generalization of these two important classes of graphs. Indeed, an interesting connection among the classes of trees, hypercubes and median graphs was proposed as a 'meta-conjecture' in 1990 [24] by the first author:

Meta-conjecture (*[24]*)**.** *Any 'reasonable' property shared by trees and hypercubes is shared by all median graphs.*

Recently, the present authors succeeded in establishing that (A), (B) and (C) characterize the median function on hypercubes, see [27]. The reader should note that the proof techniques in [27] were quite specific for hypercubes and could not be generalized. A tree being trivially a cube-free median graph, it follows from the results in [18] that (A), (B) and (C) characterize the median function on trees. Hence, one point of view is that the main result of the present paper substantiates this meta-conjecture.

The results on even profiles established in [18] are interesting in their own right, as they shed light on the structure of median graphs. To establish our main result, our first approach was to prove some more nice and surprising results for