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Locally monotone Boolean and pseudo-Boolean functions

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ABSTRACT

We propose local versions of monotonicity for Boolean and pseudo-Boolean functions: say that a pseudo-Boolean (Boolean) function is p-locally monotone if none of its partial derivatives changes in sign on tuples which differ in less than p positions. As it turns out, this parameterized notion provides a hierarchy of monotonicities for pseudo-Boolean (Boolean) functions.

Local monotonicities are shown to be tightly related to lattice counterparts of classical partial derivatives via the notion of permutable derivatives. More precisely, p-locally monotone functions are shown to have p-permutable lattice derivatives and, in the case of symmetric functions, these two notions coincide. We provide further results relating these two notions, and present a classification of p-locally monotone functions, as well as of functions having p-permutable derivatives, in terms of certain forbidden "sections", i.e., functions which can be obtained by substituting constants for variables. This description is made explicit in the special case when $p = 2$.

1. Introduction

Throughout this paper, let $[n] = \{1, \ldots, n\}$ and $\mathbb{B} = \{0, 1\}$. We are interested in the so-called Boolean functions $f : \mathbb{B}^n \to \mathbb{B}$ and pseudo-Boolean functions $f : \mathbb{B}^n \to \mathbb{R}$, where n denotes the arity of f. The pointwise ordering of functions is denoted by $\leq$, i.e., $f \leq g$ means that $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{B}^n$. The negation of $x \in \mathbb{B}$ is defined by $\overline{x} = x \oplus 1$, where $\oplus$ stands for addition modulo 2. For $x, y \in \mathbb{B}$, we set $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$.

For $k \in [n]$, $\mathbf{x} \in \mathbb{B}^n$, and $a \in \mathbb{B}$, let $\mathbf{x}_k^a$ be the tuple in $\mathbb{B}^n$ whose i-th component is a, if $i = k$, and x_i, otherwise. We use the shorthand notation $\mathbf{x}_{jk}^{ab}$ for $(\mathbf{x}_j^a)_k^b = (\mathbf{x}_k^b)_j^a$. More generally, for $S \subseteq [n]$, $\mathbf{a} \in \mathbb{B}^n$, and $\mathbf{x} \in \mathbb{B}^S$, let $\mathbf{a}_S^{\mathbf{x}}$ be the tuple in $\mathbb{B}^n$ whose i-th component is x_i, if $i \in S$, and a_i, otherwise.

Let $i \in [n]$ and $f : \mathbb{B}^n \to \mathbb{R}$. A variable x_i is said to be *essential* in f, or that f *depends* on x_i, if there exists $\mathbf{a} \in \mathbb{B}^n$ such that $f(\mathbf{a}_i^0) \neq f(\mathbf{a}_i^1)$. Otherwise, x_i is said to be *inessential* in f. Let $S \subseteq [n]$ and $f : \mathbb{B}^n \to \mathbb{R}$. We say that $g : \mathbb{B}^S \to \mathbb{R}$ is an *S-section* of f if there exists $\mathbf{a} \in \mathbb{B}^n$ such that $g(\mathbf{x}) = f(\mathbf{a}_S^{\mathbf{x}})$ for all $\mathbf{x} \in \mathbb{B}^S$. By a *section* of f we mean an S-section of f for some $S \subseteq [n]$, i.e., any function which can be obtained from f by replacing some of its variables by constants.

The *(discrete) partial derivative* of $f : \mathbb{B}^n \to \mathbb{R}$ with respect to its k-th variable is the function $\Delta_k f : \mathbb{B}^n \to \mathbb{R}$ defined by $\Delta_k f(\mathbf{x}) = f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0)$; see [9,12]. Note that $\Delta_k f$ does not depend on its k-th variable, hence it could be regarded as a function of arity $n - 1$, but for notational convenience we define it as an n-ary function.

A pseudo-Boolean function $f : \mathbb{B}^n \to \mathbb{R}$ can always be represented by a multilinear polynomial of degree at most n (see [13]), that is,

$$f(\mathbf{x}) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i, \tag{1}$$

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where $a_S \in \mathbb{R}$. For instance, the multilinear expression for a binary pseudo-Boolean function is given by

$$a_0 + a_1 x_1 + a_2 x_2 + a_{12} x_1 x_2. \tag{2}$$

This representation is very convenient for computing the partial derivatives of f. Indeed, $\Delta_k f$ can be obtained by applying the corresponding formal derivative to the multilinear representation of f. Thus, from (1), we immediately obtain

$$\Delta_k f(\mathbf{x}) = \sum_{S \ni k} a_S \prod_{i \in S \setminus \{k\}} x_i. \tag{3}$$

We say that f is *isotone* (resp. *antitone*) *in its k-th variable* if $\Delta_k f(\mathbf{x}) \geq 0$ (resp. $\Delta_k f(\mathbf{x}) \leq 0$) for all $\mathbf{x} \in \mathbb{B}^n$. If f is either isotone or antitone in its k-th variable, then we say that f is *monotone in its k-th variable*. If f is isotone (resp. antitone, monotone) in all of its variables, then f is an *isotone* (resp. *antitone*, *monotone*) *function*.[1] It is clear that any section of an isotone (resp. antitone, monotone) function is also isotone (resp. antitone, monotone). Thus defined, a function $f: \mathbb{B}^n \to \mathbb{R}$ is monotone if and only if none of its partial derivatives changes in sign on $\mathbb{B}^n$.

Noteworthy examples of monotone functions include the so-called *pseudo-polynomial functions* [2,3] which play an important role, for instance, in the qualitative approach to decision making; for general background see, e.g., [1,6]. In the current setting, pseudo-polynomial functions can be thought of as compositions $p \circ (\varphi_1, \ldots, \varphi_n)$ of (lattice) polynomial functions $p: [a, b]^n \to [a, b]$, $a < b$, with unary functions $\varphi_i: \mathbb{B} \to [a, b]$, $i \in [n]$. Interestingly, pseudo-polynomial functions $f: \mathbb{B}^n \to \mathbb{R}$ coincide exactly with those pseudo-Boolean functions that are monotone.

Theorem 1. *A pseudo-Boolean function is monotone if and only if it is a pseudo-polynomial function.*

Proof. Clearly, every pseudo-polynomial function is monotone. For the converse, suppose that $f: \mathbb{B}^n \to \mathbb{R}$ is monotone and let $a \in \mathbb{R}$ be the minimum and $b \in \mathbb{R}$ the maximum of f. Constant functions are obviously pseudo-polynomial functions, therefore we assume $a < b$. Define $\varphi_i: \mathbb{B} \to \{a, b\}$ by $\varphi_i(0) = a$ and $\varphi_i(1) = b$ if f is isotone in its i-th variable and $\varphi_i(0) = b$ and $\varphi_i(1) = a$ otherwise. Let $p: \{a, b\}^n \to [a, b]$ be given by $p = f \circ (\varphi_1^{-1}, \ldots, \varphi_n^{-1})$. Thus defined, p is isotone (i.e., order-preserving) in each variable and hence, by Theorem D in [10, p. 237], there exists a polynomial function $p': [a, b]^n \to [a, b]$ such that $p'|_{\{a,b\}^n} = p$. Therefore f is the pseudo-polynomial function $p' \circ (\varphi_1, \ldots, \varphi_n)$. □

In the special case of Boolean functions, monotone functions are most frequent among functions of small (essential) arity. For instance, among binary functions $f: \mathbb{B}^2 \to \mathbb{B}$, there are exactly two non-monotone functions, namely the Boolean sum $x_1 \oplus x_2$ and its negation $x_1 \oplus x_2 \oplus 1$. Each of these functions is in fact highly non-monotone in the sense that any of its partial derivatives changes in sign when negating its unique essential variable; this is not the case, e.g., with $f(x_1, x_2, x_3) = x_1 - x_1 x_2 + x_2 x_3$ which is non-monotone but none of its partial derivatives changes in sign when negating any of its variables (see Example 6).

This fact motivates the study of these "skew" functions, i.e., these highly non-monotone functions. To formalize this problem we propose the following parameterized relaxations of monotonicity: a function $f: \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if none of its partial derivatives changes in sign when negating less than p of its variables, or equivalently, on tuples which differ in less than p positions. With this terminology, our problem reduces to asking which Boolean functions are not 2-locally monotone. As we will see (Corollary 10), these are precisely those functions that have the Boolean sum or its negation as a binary section.

In this paper we extend this study to pseudo-Boolean functions and show that these parameterized relaxations of monotonicity are tightly related to the following lattice versions of partial derivatives. For $f: \mathbb{B}^n \to \mathbb{R}$ and $k \in [n]$, let $\wedge_k f: \mathbb{B}^n \to \mathbb{R}$ and $\vee_k f: \mathbb{B}^n \to \mathbb{R}$ be the *partial lattice derivatives* defined by

$$\wedge_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1) \quad \text{and} \quad \vee_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1).$$

The latter, known as the k-th *join derivative* of f, was proposed by Fadini [7] while the former, known as the k-th *meet derivative* of f, was introduced by Thayse [16]. In [17] these lattice derivatives were shown to be related to so-called prime implicants and implicates of Boolean functions which play an important role in the consensus method for Boolean and pseudo-Boolean functions. For further background and applications see, e.g., [4,5,8,15,18].

Observe that, just like in the case of the partial derivative $\Delta_k f$, the k-th variable of each of the lattice derivatives $\wedge_k f$ and $\vee_k f$ is inessential.

The following proposition assembles some basic properties of lattice derivatives.

Proposition 2. *For any pseudo-Boolean functions $f, g: \mathbb{B}^n \to \mathbb{R}$ and $j, k \in [n]$, $j \neq k$, the following hold:*

(i) $\wedge_k \wedge_k f = \wedge_k f$ *and* $\vee_k \vee_k f = \vee_k f$;
(ii) *if* $f \leq g$, *then* $\wedge_k f \leq \wedge_k g$ *and* $\vee_k f \leq \vee_k g$;
(iii) $\wedge_j \wedge_k f = \wedge_k \wedge_j f$ *and* $\vee_j \vee_k f = \vee_k \vee_j f$;
(iv) $\vee_k \wedge_j f \leq \wedge_j \vee_k f$.

[1] Note that the terms "positive" and "nondecreasing" (resp. "negative" and "nonincreasing") are often used instead of isotone (resp. antitone), and it is also customary to use the word "monotone" only for isotone functions.