Discrete Applied Mathematics 160 (2012) 1757–1765

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Hypergraphs with large domination number and with edge sizes at least three

Michael A. Henning, Christian Löwenstein *

Department of Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa

ARTICLE INFO

Article history:
Received 13 September 2011
Received in revised form 16 March 2012
Accepted 20 March 2012
Available online 20 April 2012

Keywords:
Domination
Transversal
Hypergraph

ABSTRACT

Let $H = (V, E)$ be a hypergraph with vertex set V and edge set E. A dominating set in H is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. The domination number $\gamma(H)$ is the minimum cardinality of a dominating set in H. It is known that if H is a hypergraph of order n with edge sizes at least three and with no isolated vertex, then $\gamma(H) \leq n/3$. In this paper, we characterize the hypergraphs achieving equality in this bound.

1. Introduction

In this paper we continue the study of domination in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*. We shall use the notation $n_H = |V|$ and $m_H = |E|$, and sometimes simply n and m without subscript if the actual H need not be emphasized, to denote the order and size of H, respectively. In the problems studied here, one may assume that $|e| \geq 2$ holds for all $e \in E$, and it will also be possible to avoid multiple edges without loss of generality.

A *k-edge* in H is an edge of size k. The hypergraph H is said to be *k-uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The hypergraph H is *simple* if no two edges contain exactly the same vertex set. The *degree* of a vertex v in H, denoted by $d_H(v)$ or simply by $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The minimum degree among the vertices of H is denoted by $\delta(H)$. Two edges in H are said to be *overlapping* if they intersect in at least two vertices. We define a hypergraph H to be *edge-minimal* if every edge of H contains at least one vertex of degree 1 in H. If H' is a hypergraph such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$, then H' is called a *subhypergraph* of H and we write $H' \subset H$. Possibly, $H' = H$. Further if $V(H') = V(H)$, then H' is called a *spanning subhypergraph* of H.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. Further, x and y are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *connected component* or simply a *component* of H. Thus, no edge in H contains vertices from different components.

If H denotes a hypergraph and X denotes a subset of vertices in H, then $H - X$ will denote that hypergraph obtained by removing the vertices X from H and removing all hyperedges that intersect X. We remark that in the literature this is sometimes denoted by *strongly deleting* the vertices in X.

* Corresponding author. Fax: +27 11 559 2874.
E-mail addresses: mahenning@uj.ac.za (M.A. Henning), christian.loewenstein@uni-ulm.de (C. Löwenstein).

0166-218X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2012.03.023

If $H = (V, E)$ denotes a hypergraph and $E' \subseteq E$, then by the hypergraph induced (or formed) by the edges in E' we mean the hypergraph with edge set E' and with vertex set consisting of all vertices that belong to some edge in E'. The hypergraph $H - E'$ denotes the hypergraph with vertex set V and edge set $E \setminus E'$ that is obtained from H by deleting all edges in E'.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H.

A *dominating set* in a hypergraph $H = (V, E)$ is a subset of vertices $D \subseteq V$ such that for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ for which $v \in e$ and $e \cap D \neq \emptyset$. Equivalently, every vertex $v \in V \setminus D$ is adjacent with a vertex in D. The *domination number* $\gamma(H)$ is the minimum cardinality of a dominating set in H. A dominating set of H of cardinality $\gamma(H)$ is called a $\gamma(H)$-set.

If $H = (V, E)$ is a hypergraph and $X, Y \subseteq V$, then we say that Y *dominates* X if every vertex in X is in Y or is adjacent to some vertex of Y. If X is a nonempty subset of vertices in H, then we define an *X-dominating set* in H as a set Y that dominates X and we define the *X-domination number*, denoted $\gamma(X; H)$ as the minimum cardinality of an X-dominating set in H. In particular, we note that $\gamma(H) = \gamma(V; H)$.

Domination in graphs is still a very active area inside graph theory; see, for example, the recent papers [7,10,12,15]. Domination in hypergraphs, however, was introduced recently by Acharya [1] and studied further, for example, in [2–4,11].

1.1. Known results on transversal and domination in hypergraphs

Transversals in hypergraphs are well studied in the literature. We shall need the following upper bound on the transversal number of a hypergraph in terms of its order and size.

Theorem 1 (*Erdős and Tuza [6]*). *If H is a connected hypergraph on n vertices and m edges with all edges of size at least two, then $\tau(H) \leq 2(n + m + 1)/7$.*

Chvátal and McDiarmid [5] and Tuza [14] independently established that if H is a 3-uniform hypergraph on n vertices and m edges, then $4\tau(H) \leq n + m$. We remark that a short proof of this result can be found in [13]. The extremal hypergraphs that achieve equality in this bound were characterized in [9].

As a special case of a result due to Bujtás et al. [4], we have the following upper bound on the domination number of a 3-uniform hypergraph in terms of its order. We remark that this result can also be deduced from a result in [8] which states that if every edge in a graph G with no isolated vertex and of order n is contained in a triangle, then $\gamma(G) \leq n/3$.

Theorem 2 (*[4,8]*). *If H is a hypergraph of order n with all edges of size at least three and with no isolated vertex, then $\gamma(H) \leq n/3$.*

Our aim in this paper is to characterize the hypergraphs achieving equality in the upper bound of Theorem 2.

1.2. The families $\mathcal{H}$ and $\mathcal{H}_3$

Let $H_1, H_2, \ldots, H_{15}$ be the fifteen hypergraphs shown in Fig. 1. Let H_{under} be a hypergraph every component of which is isomorphic to a hypergraph H_i for some i, $1 \leq i \leq 15$. Each component of H_{under} we call a *unit* of H_{under}. In each unit we 2-color the vertices with the colors black and white as indicated in Fig. 1 and we call the white vertices the *link vertices* of the unit and the black vertices the *non-link vertices*.

Let H be a hypergraph obtained from H_{under} by adding edges of size at least three, called *link edges*, in such a way that every added edge contains vertices from at least two units and contains only link vertices. Possibly, H is disconnected or $H = H_i$ for some i, $1 \leq i \leq 15$. We call the hypergraph H_{under} an *underlying hypergraph* of H and we let $\mathcal{U}(H_{\text{under}})$ denote the set of all units in H_{under}. Let $\mathcal{H}$ denote the family of all such hypergraphs H and let $\mathcal{H}_3$ denote the subfamily of $\mathcal{H}$ consisting of all 3-uniform hypergraphs in $\mathcal{H}$.

2. Main result

In this paper, we characterize the hypergraphs with no isolated vertex and with all edges of size at least three whose domination number is one-third their order. We shall prove the following result.

Theorem 3. *Let H be a hypergraph of order n with all edges of size at least three and with $\delta(H) \geq 1$. Then, $\gamma(H) \leq n/3$ with equality if and only if $H \in \mathcal{H}$.*

2.1. Preliminary observations and lemmas

Since every transversal in a hypergraph with no isolated vertex is a dominating set in the hypergraph, we have the following observation.

Observation 4. If H is a hypergraph with $\delta(H) \geq 1$, then $\gamma(H) \leq \tau(H)$.