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Collapsible graphs and Hamiltonian connectedness of line graphs[☆]

Weihua Yang[a,*], Hongjian Lai[b,c], Hao Li[a], Xiaofeng Guo[d]

[a] Laboratoire de Recherche en Informatique, UMR 8623, C.N.R.S.-Université de Paris-sud, 91405-Orsay cedex, France

[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

[c] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, China

[d] School of Mathematical Science, Xiamen University, Xiamen Fujian 361005, China

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ABSTRACT

Thomassen conjectured that every 4-connected line graph is Hamiltonian. Chen and Lai [Z.-H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics—an update, in: Ku Tung-Hsin (Ed.), Combinatorics and Graph Theory, vol. 95, World Scientific, Singapore/London, 1995, pp. 53–69, Conjecture 8.6] conjectured that every 3-edge connected, essentially 6-edge connected graph is collapsible. In this paper, we prove the following results. (1) Every 3-edge connected, essentially 6-edge connected graph with edge-degree at least 7 is collapsible. (2) Every 3-edge connected, essentially 5-edge connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible which implies that 5-connected line graph with minimum degree at least 6 of a graph with at most 24 vertices of degree 3 is Hamiltonian. (3) Every 3-connected, essentially 11-connected line graph is Hamilton-connected which strengthens the result in [H.-J. Lai, Y. Shao, H. Wu, J. Zhou, Every 3-connected, essentially 11-connected line graph is Hamiltonian, J. Combin. Theory, Ser. B 96 (2006) 571–576] by Lai et al. (4) Every 7-connected line graph is Hamiltonian connected which is proved by a method different from Zhan's. By using the multigraph closure introduced by Ryjáček and Vrána which turns a claw-free graph into the line graph of a multigraph while preserving its Hamilton-connectedness, the results (3) and (4) can be extended to claw-free graphs.

1. Introduction

Unless stated otherwise, we follow [1] for terminology and notations, and we consider finite connected graphs without loop. In particular, we use $\kappa(G)$ and $\kappa'(G)$ to represent the *connectivity* and *edge-connectivity* of a graph G. A graph is *trivial* if it contains no edges. A vertex cut X of G is *essential* if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-connected* if G does not have an *essential cut* X with $|X| < k$. An edge cut Y of G is *essential* if $G - Y$ has at least two non-trivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$. In particular, the *essential edge-connectivity* of G, denote by $\lambda'(G)$, is the size of a minimum essential edge-cut. Let $u \in V(G)$ and $d_G(u)$ the degree of u, or simply $d(u)$ if no confusion. For $e = uv \in E(G)$, define $d(e) = d(u) + d(v) - 2$ the edge degree of e, and $\xi(G) = \min\{d(e) : e \in E(G)\}$. Esfahanian in [6] proved that if a connected graph G with $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ exists and $\lambda'(G) \leq \xi(G)$. Thus, a essentially k-edge connected graph has edge-degree at least k. Denote $D_i(G)$ and $d_i(G)$ the set of vertices of degree i and $|D_i(G)|$, respectively. If no confusion, we directly use D_i and d_i for $D_i(G)$ and $d_i(G)$, respectively. For a subgraph $A \subseteq G$, $v \in V(G)$, $N_G(v)$ denotes

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* Corresponding author. Fax: +33 86 991 8585505.
E-mail address: ywh222@163.com (W. Yang).

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the set of the neighbors of v in G and $N_G(A)$ denotes the set $\left(\bigcup_{v\in V(A)} N_G(v)\right) \setminus V(A)$. If no confusion arises, we use an edge uv for a subgraph with three elements of $\{u, v, uv\}$. Denote $G[X]$ the subgraph induced by the vertex set X of $V(G)$.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. Thomassen in 1986 posed the following conjecture:

Conjecture 1.1 (*Thomassen [18]*). *Every 4-connected line graph is Hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a claw-free graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture:

Conjecture 1.2 (*Matthews and Sumner [14]*). *Every 4-connected claw-free graph is Hamiltonian.*

Theorem 1.3 (*Zhan [20]*). *Every 7-connected line graph is Hamiltonian connected.*

Theorem 1.4 (*Ryjáček [15]*).

(i) *Conjectures* 1.1 and 1.2 *are equivalent.*
(ii) *Every 7-connected claw-free graph is Hamiltonian.*

Very recently, an important progress towards the conjectures was submitted by Kaiser and Vrána [9] in which the following theorem is listed:

Theorem 1.5 (*[9]*). *Every 5-connected line graph with minimum degree at least 6 is Hamiltonian.*

So we clearly have:

Corollary 1.6. *Every 6-connected line graph is Hamiltonian.*

Using Ryjáček's line graph closure, the following corollary is obtained:

Corollary 1.7 (*[9]*). *Every 5-connected claw-free graph G with minimum degree at least 6 is Hamiltonian.*

We list some known partial results on the Hamiltonicity of line graphs and claw-free graphs as follows. Chen et al. in [5] reported that every 4-connected line graph $L(G)$ with $D_3(G) = \emptyset$ is Hamiltonian. Li in [13] proved that every 6-connected claw-free graph with at most 33 vertices of degree 6 is Hamiltonian. Let G be a 6-connected line graph. Hu et al. in [7] showed that if $d_6(G) \leq 29$ or $G[D_6(G)]$ contains at most 5 vertex disjoint K_4's, then G is Hamilton-connected. Let G be a 6-connected claw-free graph. Hu et al. in [8] showed that if $d_6(G) \leq 44$ or $G[D_6(G)]$ contains at most 8 vertex disjoint K_4's, then G is Hamiltonian. Let G be a 6-connected line graph. Zhan in [21] showed that if either $d_6(G) \leq 74$, or $d_6(G) \leq 54$ or $G[D_6(G)]$ contains at most 5 vertex disjoint K_4's, then G is Hamiltonian.

In particular, Lai et al. in [11] considered the following problem: For 3-connected line graphs, can high essential connectivity guarantee the existence of a Hamiltonian cycle? They proved the following theorem:

Theorem 1.8 (*Lai et al. [11]*). *Every 3-connected, essentially 11-connected line graph is Hamiltonian.*

We shall prove that every 3-connected, essentially 11-connected line graph is Hamilton-connected in Section 4.

Chen and Lai in [4] posed the following conjectures:

Conjecture 1.9 (*Chen and Lai Conjecture 8.6 [4]*). *Every 3-edge connected, essentially 6-edge connected graph G is collapsible.*

Conjecture 1.10 (*Chen and Lai Conjecture 8.7 [4]*). *Every 3-edge connected, essentially 5-edge connected graph G is super-Eulerian.*

Now we list the results of the current paper. We prove that (1) Every 3-edge connected, essentially 6-edge connected graph with edge-degree at least 7 is collapsible. (2) Every 3-edge connected, essentially 5-edge connected graph with edge-degree at least 6 and at most 24 vertices of degree 3 is collapsible which implies that 5-connected line graph with minimum degree at least 6 of a graph with at most 24 vertices of degree 3 is Hamiltonian. (3) Every 3-connected, essentially 11-connected line graph is Hamilton-connected which strengthens the result of Lai et al. in [11]. (4) Every 7-connected line graph is Hamilton-connected which is proved by a method different from Zhan's [20].