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On plane graphs with link component number equal to the nullity

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ABSTRACT

In this paper, we study connected plane graphs with link component number equal to the nullity and call them near-extremal graphs. We first study near-extremal graphs with minimum degree at least 3 and prove that a connected plane graph G with minimum degree at least 3 is a near-extremal graph if and only if G is isomorphic to K_4, the complete graph with 4 vertices. The result is obtained by studying general graphs using the knowledge of bicycle space and the Tutte polynomial. Then a simple algorithm is given to judge whether a connected plane graph is a near-extremal graph or not. Finally we study the construction of near-extremal graphs and prove that all near-extremal graphs can be constructed from a loop and K_4 by two graph operations.

1. Introduction

Let G be a graph. We denote by $p(G)$, $q(G)$ and $k(G)$ numbers of vertices, edges and connected components of the graph G, respectively. The *nullity* $n(G)$ of the graph G is defined to be $q(G) - p(G) + k(G)$. When G is a connected plane graph, $n(G)$ is equal to the number of bounded faces of the plane graph G by the well-known Euler formula.

Let $T_G(x, y)$ be the Tutte polynomial [15] of the graph G. Let

$$\mu(G) = \log_2(|T_G(-1, -1)|) + k(G).$$

We call $\mu(G)$ the *link component number* of the graph G, since when G is a plane graph, $\mu(G)$ is exactly the number of components of the link corresponding to G via the classical medial construction. See [8,9,11,13].

In [7], the authors proved that $1 \le \mu(G) \le n(G) + 1$ for any connected plane graph G and characterized *extremal graphs*, i.e connected plane graphs with $\mu(G) = n(G) + 1$. A family of oriented links which correspond to extremal graphs was once considered in [5]. Connected plane graphs with $\mu(G) = 1$ (i.e. knot graphs) were studied forty years ago; see [14]. The structure of such knot graphs was studied in [3]. The component number of links corresponding to some families of plane graphs has been determined. See, for example, [10,12,6].

In this paper, we shall study connected plane graphs with $\mu(G) = n(G)$ and call them *near-extremal graphs*. We are mainly interested in plane graphs because of its connection to alternating links. Note that different embeddings in the plane of a planar graph may correspond to different alternating links with the same component numbers. See Fig. 1 for such an example.

Let G_1 and G_2 be two graphs. We use $G_1 \cong G_2$ to denote that G_1 is isomorphic to G_2. We first study near-extremal graphs with minimum degree at least 3 and prove that a connected plane graph G with minimum degree at least 3 is a near-extremal

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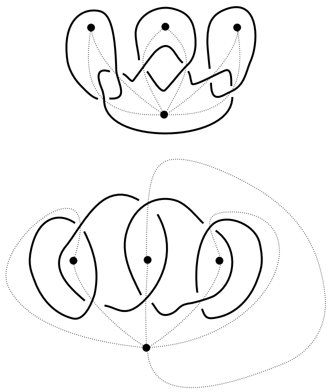

Fig. 1. Two different plane embeddings of a planar graph correspond to two different alternating links.

graph if and only if $G \cong K_4$, the complete graph with 4 vertices. The result is obtained by studying general graphs using the knowledge of bicycle space and the Tutte polynomial [4]. Then a simple algorithm is given to judge whether a connected plane graph is a near-extremal graph or not. Finally we study the construction of near-extremal graphs and show that all near-extremal graphs can be constructed from a loop and K_4 by two graph operations.

We point out that the components of the link formed from a plane graph correspond to the left–right paths (Petrie circuits) of the plane graph, which play an important role in the design of CMOS VLSI circuits; see references cited in [16].

Throughout the paper, we denote by C_n the n-cycle. In particular, C_1 is a loop. We denote by I_n the graph which consists of two distinct vertices connected by n parallel edges. We use K_n to denote the complete graph with n vertices. Let G be a graph and e be an edge of G. We use $G - e$ and G/e to denote the graphs obtained from G by deleting and contracting (that is, deleting the edge and identifying its ends) the edge e, respectively. We follow [2] for undefined terminology and notations.

2. Preliminary results

Let G be a general graph and let $C(G)$ and $C^*(G)$ be its cycle space and cut space respectively. Then its bicycle space $B(G)$ is defined to be $C(G) \cap C^*(G)$. Let $b(G)$ denote the dimension of $B(G)$. Then since the dimension of the cycle space is $n(G)$ (see [4]), we have the following.

Lemma 2.1. *For any graph G, $b(G) \leq n(G)$.*

We say that a general graph G is *G-extremal* if $b(G) = n(G)$ and *G-near-extremal* if $b(G) = n(G) - 1$. Then G is extremal if and only if G is G-extremal and a connected plane graph, and G is near-extremal if and only if G is G-near-extremal and a connected plane graph.

Using the fact that $b(G) = \log_2(|T_G(-1, -1)|) = \mu(G) - k(G)$ [13,11] and properties of the Tutte polynomial [1], we can prove Lemmas 2.2–2.4. The proofs are similar to the proof of Theorem 4.1 in [11], so we omit the details here.

Lemma 2.2. *Let G_i be a graph with $v_i \in V(G_i)$ for $i = 1, 2$. Let G be the graph obtained from G_1 and G_2 by identifying v_1 and v_2. Then $b(G) = b(G_1) + b(G_2)$. In particular,*

(1) *if e is a loop, then $b(G) = b(G - e)$;*
(2) *if e is a bridge, then $b(G) = b(G/e)$.*

A pair of edges of G is called a *parallel pair* if the pair of edges has the same endvertices; a pair of edges of G is called a *series pair* if it is not a parallel pair and both edges are incident with the same vertex of degree 2.

Lemma 2.3. *Let G be a graph. Then:*

(1) *if e and f are a series pair of G, then $b(G/e/f) = b(G)$;*
(2) *if e and f are a parallel pair of G and f is not a bridge in $G - e$, then $b(G - e - f) = b(G)$.*

Let G be a graph with a vertex v of degree 3, which has three distinct neighbors. Let H be the graph obtained from G by deleting v and adding an edge between each pair of neighbors of v. We say that H is obtained from G by a *$Y\Delta$-exchange* and conversely, G is obtained from H by a *ΔY-exchange*.

Lemma 2.4. *Suppose that G' is obtained from G by either a $Y\Delta$ or a ΔY exchange. Then $b(G) = b(G')$.*

Lemma 2.5. *If G has an odd cycle, then G is not G-extremal.*