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Equitable colorings of Cartesian products of graphs☆

Wu-Hsiung Lin [a], Gerard J. Chang [a,b,c,*]

[a] Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

[b] Taida Institute for Mathematical Sciences, National Taiwan University, Taipei 10617, Taiwan

[c] National Center for Theoretical Sciences, Taipei Office, Taiwan

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ABSTRACT

The present paper studies the following variation of vertex coloring on graphs. A graph G is equitably k-colorable if there is a mapping $f: V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for $xy \in E(G)$ and $\| f^{-1}(i) | - | f^{-1}(j) \| \leq 1$ for $1 \leq i, j \leq k$. The equitable chromatic number of a graph G, denoted by $\chi_=(G)$, is the minimum k such that G is equitably k-colorable. The equitable chromatic threshold of a graph G, denoted by $\chi_=^*(G)$, is the minimum t such that G is equitably k-colorable for all $k \geq t$. Our focus is on the equitable colorability of Cartesian products of graphs. In particular, we give exact values or upper bounds of $\chi_=(G\Box H)$ and $\chi_=^*(G\Box H)$ when G and H are cycles, paths, stars, or complete bipartite graphs.

1. Introduction

Graph coloring plays a central position in discrete mathematics. During the past hundred years, many deep and interesting results have been obtained, and various applications have arisen. In the current paper, we focus on a restricted version of graph coloring called equitable coloring.

For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$. A (*proper*) *k-coloring* of a graph G is a mapping $f: V(G) \to [k]$ such that $f(x) \neq f(y)$ for $xy \in E(G)$. We call the set $f^{-1}(i) = \{x \in V(G): f(x) = i\}$ a *color class* for $i \in [k]$. Notice that each color class is an independent set, i.e., a pairwise non-adjacent vertex set. A graph is *k-colorable* if it has a k-coloring. The *chromatic number* of G is $\chi(G) = \min\{k: G \text{ is } k\text{-colorable}\}$.

This paper focuses on the following variation of coloring. An *equitable k-coloring* is a k-coloring for which any two color classes differ in size by at most 1. If a graph of n vertices is equitably k-colorable then each color class is of size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$; more precisely, the color classes have sizes $\lfloor \frac{n+i-1}{k} \rfloor \left(= \lceil \frac{n-k+i}{k} \rceil\right)$ for $i \in [k]$. The *equitable chromatic number* of G is $\chi_=(G) = \min\{k: G \text{ is equitably } k\text{-colorable}\}$ and the *equitable chromatic threshold* of G is $\chi_=^*(G) = \min\{t: G \text{ is equitably } k\text{-colorable for all } k \geq t\}$. The concept of equitable colorability was first introduced by Meyer [26]. His motivation came from the application given by Tucker [32] where vertices represented garbage collection routes and two such vertices were joined when the corresponding routes should not be run on the same day. For more applications such as scheduling and constructing timetables, please see [1,12,13,16,28,31,32]. For a good survey, please see the paper by Lih [23].

In 1964 Erdős [7] conjectured that any graph G with maximum degree $\Delta(G) \leq k$ has an equitable $(k+1)$-coloring. This conjecture was proved in 1970 by Hajnal and Szemerédi [9] with a long and complicated proof. Mydlarz and Szemerédi [27] found a polynomial-time algorithm for such a coloring. Recently, Kierstead and Kostochka [14] gave a short proof of the

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* Corresponding author at: Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan. Tel.: +886 2 3366 2863; fax: +886 2 2367 5981.

E-mail addresses: d92221001@ntu.edu.tw (W.-H. Lin), gjchang@math.ntu.edu.tw (G.J. Chang).

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theorem, and presented another polynomial-time algorithm for such a coloring. They [15] proved an even stronger result that every graph satisfying $d(x) + d(y) \le 2k + 1$ for every edge xy has an equitable $(k + 1)$-coloring. Brooks type results are conjectured: the Equitable Coloring Conjecture [26] $\chi_=(G) \le \Delta(G)$, and the Equitable Δ-Coloring Conjecture [5] $\chi^*_=(G) \le \Delta(G)$ for $G \notin \{K_n, C_{2n+1}, K_{2n+1,2n+1}\}$. Exact values of equitable chromatic numbers and equitable thresholds of trees [3,4] and complete multipartite graphs [2,22] were determined. Chen et al. [6] and Furmańczyk [8] investigated equitable colorability of square and cross products of graphs. Equitable coloring has been extensively studied in the literature; see [4,5,17–21, 23–25,28,29,34–36].

Among the known results on equitable coloring, we are most interested in those on graph products. Notice that studying the relation of graph parameters between the product and its factors is helpful for analyzing the structure of complicated graphs; see [10,11,30,33,37]. The *Cartesian* (or *square*) *product* of graphs G and H is the graph $G\Box H$ with vertex set $\{(x, y): x \in V(G), y \in V(H)\}$ and edge set $\{(x, y)(x', y'): x = x'$ with $yy' \in E(H)$ or $xx' \in E(G)$ with $y = y'\}$.

This paper is organized as follows. Section 2 is a review for equitable colorings on Cartesian products of graphs related to our results in this paper. Section 3 establishes exact values of equitable chromatic numbers and thresholds of Cartesian products of an odd cycle or an odd path with a bipartite graph, an even cycle or an even path with a complete bipartite graph, and two stars; and upper bounds on the equitable chromatic number and threshold of the Cartesian product of two complete bipartite graphs. In the last section, we summarize our results and give some open problems.

2. Preliminaries

For an integer positive n, the *n-path* is the graph P_n with vertex set $\{x_1, x_2, \ldots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. For an integer $n \ge 3$, the *n-cycle* is the graph C_n with vertex set $\{x_1, x_2, \ldots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$. For positive integers m and n, the *complete bipartite graph* $K_{m,n}$ is the graph with vertex set $\{y_i, z_j : 1 \le i \le m, 1 \le j \le n\}$ and edge set $\{y_iz_j : 1 \le i \le m, 1 \le j \le n\}$. A *bipartite graph* is a subgraph of a complete bipartite graph.

It is evident from the definition that $\chi(G) \le \chi_=(G) \le \chi^*_=(G)$ for any graph G. In general, the inequalities can be strict. For example,

$$\chi(K_{1,4}) = 2 < \chi_=(K_{1,4}) = \chi^*_=(K_{1,4}) = 3,$$
$$\chi(K_{3,3}) = \chi_=(K_{3,3}) = 2 < \chi^*_=(K_{3,3}) = 4,$$
$$\chi(K_{5,8}) = 2 < \chi_=(K_{5,8}) = 3 < \chi^*_=(K_{5,8}) = 5.$$

The following result by Chen et al. [6] is of most interest in our study on equitable colorability for Cartesian products of graphs.

Theorem 1 (*[6]*). *If G and H are equitably k-colorable, then so is $G\Box H$.*

Consequently, we have the following inequality for the equitable chromatic threshold:

Corollary 2. $\chi^*_=(G\Box H) \le \max\{\chi^*_=(G), \chi^*_=(H)\}$.

Corollary 3. *If G and H are graphs with $\chi(G) = \chi^*_=(G)$ and $\chi(H) = \chi^*_=(H)$, then $\chi(G\Box H) = \chi_=(G\Box H) = \chi^*_=(G\Box H) = \max\{\chi(G), \chi(H)\}$.*

Proof. The result follows from Corollary 2 and Sabidussi's result [30] that $\chi(G\Box H) = \max\{\chi(G), \chi(H)\}$. $\Box$

Examples of graphs G with $\chi(G) = \chi^*_=(G)$ include complete graphs, paths and cycles; see [6,8] for Corollary 3 on these three classes of graphs. For instance, $\chi(K_m\Box K_n) = \chi_=(K_m\Box K_n) = \chi^*_=(K_m\Box K_n) = \max\{m, n\}$, $\chi(C_m\Box C_n) = \chi_=(C_m\Box C_n) = \chi^*_=(C_m\Box C_n) = 2$ (resp. 3) if m and n are even (resp. m or n is odd), and $\chi_=(K_{1,m}\Box P_n) = 3$ for $m \ge 3$ and odd $n \ge 3$.

Unlike the equitable chromatic threshold, $\chi_=(G\Box H) \le \max\{\chi_=(G), \chi_=(H)\}$ is false in general. For instance, Chen et al. [6] showed that $\chi_=(K_{1,1,2}\Box K_{3,3}) = 4 > \max\{\chi_=(K_{1,1,2}), \chi_=(K_{3,3})\} = 3$. They [6] also mentioned that $\chi_=(G) = \chi_=(H) = k$ may not lead to $\chi_=(G\Box H) = k$ with the example $\chi_=(K_{1,2n}) = n + 1$ while $\chi_=(K_{1,2n}\Box K_{1,2n}) \le 4$. In Section 3 we shall give more general results of this kind.

3. The Cartesian product of graphs

We now study equitable chromatic numbers and equitable chromatic thresholds of Cartesian products of graphs for three cases as follows.

3.1. The product of $C_{2\ell+1}$ or $P_{2\ell+1}$ with a bipartite graph

We first study the Cartesian product of an odd cycle or an odd path with a bipartite graph.

Theorem 4. *If ℓ is a positive integer and H is a bipartite graph, then $\chi_=(C_{2\ell+1}\Box H) = \chi^*_=(C_{2\ell+1}\Box H) = \chi_=(P_{2\ell+1}\Box H) = \chi^*_=(P_{2\ell+1}\Box H) = 3$ except that $\chi_=(P_{2\ell+1}\Box H) = \chi^*_=(P_{2\ell+1}\Box H) = 2$ for the case when $\chi_=(H) \le 2$.*

Proof. Recall that the vertex set of $C_{2\ell+1}$ or $P_{2\ell+1}$ is $\{x_1, x_2, \ldots, x_{2\ell+1}\}$. Suppose the bipartition of the graph H consists of $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$. We order the vertices of the product graph $C_{2\ell+1}\Box H$ or $P_{2\ell+1}\Box H$ as in Fig. 1. Notice that any set consisting of consecutive vertices in the ordering of size no more than $\ell(m + n)$ is an independent set.