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Note

Optimal stopping in a search for a vertex with full degree in a random graph☆

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ABSTRACT

We consider the following on-line decision problem. The vertices of a realization of the random graph $\mathcal{G}(n, p)$ are being observed one by one by a selector. At time m, the selector examines the mth vertex and knows the graph induced by the m vertices that have already been examined. The selector's aim is to choose the currently examined vertex maximizing the probability that this vertex has full degree, i.e. it is connected to all other vertices in the graph. An optimal algorithm for such a choice (in other words, optimal stopping time) is given. We show that it is of a threshold type and we find the threshold and its asymptotic estimation.

1. Introduction

The well known secretary problem can be stated as follows: n linearly ordered candidates for a job as a secretary arrive at an interview in some random permutation. The selector's task is to pick the very best of them, but his choice has some serious limitations. He must make his decision at some moment τ picking the presently examined candidate. What is more, his choice is based only on his knowledge of the relative ranks of the candidates examined so far and the number n of all candidates. This problem was solved in [11]. The optimal algorithm is of a threshold type, i.e. the selector must wait till a certain moment, asymptotically n/e, and then choose the first candidate that is the best up to now. The probability of success is asymptotically $1/e$.

This nice problem developed in various directions; see, for instance, the interesting surveys [2,15]. One of them consisted of considering a partial order instead of the linear one, for instance, [16,7] (a survey of a research of several authors), [8,12,13,5,4]. In [9,3] and [6], the aim of the selector is to choose in an optimal way a candidate from a given group.

The next generalization [10] came with realizing that orders are very rich directed graphs. Authors considered the problem of choosing the maximal element of a directed path (the analogue of the best candidate in the linear order case). Actually, there is no obstacle to formulate a still more general problem, where for a given graph we want to choose in the on-line decision process a vertex from some predefined set of vertices. In particular, the graph-theoretic variant of [9,3,6] was considered in [14].

In this paper, we go one step further and consider the problem of optimal choice of a vertex from a given subset of vertices of a random graph. Namely, we find an optimal algorithm for choosing a vertex of a realization of the random graph $\mathcal{G}(n, p)$ that has full degree. More information about random graphs can be found in [1].

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2. Definitions and notation

A *graph* is a pair (V, E), where V is a set of *vertices* and E is a family of nonempty subsets of V of cardinality at most two. Each such subset is called an *edge* (connecting its elements). A vertex v has *full degree* if $\{v, u\} \in E$ for every $u \in V, u \neq v$. The set of vertices of G with full degree will be denoted by $F(G)$. For a graph $G = (V, E)$, its *subgraph induced by* $W \subset V$ is the graph

$$G \restriction W = (W, E \cap \{\{w_1, w_2\} : w_1, w_2 \in W\}).$$

The random graph $\mathcal{G}(n, p)$ is a probability space, whose elements are all 2^n labeled graphs $G = (V, E)$ with vertex set $V = (v_1, \dots, v_n)$, without edges of cardinality one (loops), where each pair of vertices v_i, v_j, $1 \le i < j \le n$ is chosen to be an edge of G independently and with probability p.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \mathcal{F}_n \subseteq \mathcal{F}$ be a sequence of σ-algebras. We call such a sequence a *filtration*. We say that a random variable $\tau : \omega \to \{1, 2, \dots, n\}$ is a *stopping time* with respect to a filtration $(\mathcal{F}_t)_{t=1}^n$ if $\tau^{-1}(\{t\}) \in \mathcal{F}_t$ for each $t \le n$. We shall denote the set of all stopping times by $\mathcal{T}$.

If we think of $\tau(\omega)$, $\omega \in \Omega$, as a moment when to stop observing a certain process depending on ω and $t = 1, 2, \dots, n$, then the condition $\tau^{-1}(\{t\}) \in \mathcal{F}_t$ means that our decision to stop at t is based only on the events that took place until this moment and does not depend on any information about the future events.

For $x > 0$, we shall use $W(x)$ to denote the unique real value of the Lambert W-function at x, which is the inverse function of $f(z) = ze^z$.

Remark. In many optimal stopping problems, e.g. the one concerning linear or partial orders, the elements are assumed to arrive in some random permutation. In our problem we do not make this assumption. Because of the random character of the graph we are considering, we can simply assume that the vertices come in a given fixed order.

3. Choosing a vertex with full degree

For fixed $n \in \mathbb{N}$, $p \in (0, 1)$, suppose $G = (V, E)$ is a realization of $\mathcal{G}(n, p)$. Our goal is to choose a vertex with full degree of G (notice that such vertex may not exist in G). In the mth step of our search, we observe a vertex v_m and then construct the graph $G \restriction \{v_1, \dots, v_m\}$. It is easy to see that this graph is a realization of $\mathcal{G}(m, p)$. Let us define random variables

$$X_i(G) = \begin{cases} 1, & \text{when } v_i \text{ has full degree in } G \restriction \{v_1, \dots, v_i\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$Y_i(G) = \begin{cases} 1, & \text{when } v_i \text{ has full degree in } G, \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 1, 2, \dots, n$ let us define the following stopping times

$$\tau_k = \inf\{t \ge k : X_t = 1 \text{ or } t = n\}.$$

Let us denote the set of these stopping times by $\mathcal{T}_n$ and the set of all stopping times by $\mathcal{T}$.

Lemma 3.1. *For some $k \in \{1, 2, \dots, n\}$ the time τ_k is optimal, i.e.*

$$\mathbb{P}[v_{\tau_k} \in F(G)] = \max_{\tau \in \mathcal{T}} \mathbb{P}[v_\tau \in F(G)].$$

Proof. We should choose the ith vertex if the probability of winning with it exceeds the probability of winning with the best strategy available if we continue. If $X_i(G) = 0$, then also $Y_i(G) = 0$ and so we need only consider choosing the ith vertex if $X_i(G) = 1$. We now observe that the probability of winning with such a vertex at draw i equals p^{n-i}; thus it is a strictly increasing function of i. The probability of winning with the best strategy available if we continue is a decreasing function of i, since we can always get to a later point in the sequence and then use whatever strategy is available. Consequently, the optimum strategy is of a threshold type. It means that for some k we should draw the first $k - 1$ vertices and then choose the first vertex v_i, $i \ge k$, such that $X_i(G) = 1$. $\square$

We shall now focus on finding the optimal time in the set $\mathcal{T}_n$. Let

$$c_k = \mathbb{P}[Y_{\tau_k} = 1] = \mathbb{P}[v_{\tau_k} \in F]$$

be the probability of choosing a vertex with full degree using strategy τ_k.

Lemma 3.2.

$$c_n = p^{n-1},$$
$$c_k = p^{n-1} + \left(1 - p^{k-1}\right) c_{k+1}, \quad 1 \le k \le n - 1.$$