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The $L(2, 1)$-labeling of unigraphs

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ABSTRACT

The $L(2, 1)$-*labeling problem* consists of assigning colors from the integer set $0, \ldots, \lambda$ to the nodes of a graph G in such a way that nodes at a distance of at most two get different colors, while adjacent nodes get colors which are at least two apart. The aim of this problem is to minimize λ and it is in general NP-complete. In this paper the problem of $L(2, 1)$-labeling unigraphs, i.e. graphs uniquely determined by their own degree sequence up to isomorphism, is addressed and a $3/2$-approximate algorithm for $L(2, 1)$-labeling unigraphs is designed. This algorithm runs in $O(n)$ time, improving the time of the algorithm based on the greedy technique, requiring $O(m)$ time, that may be near to $\Theta(n^2)$ for unigraphs.

1. Introduction

The $L(2, 1)$-*labeling problem* [10] consists in assigning colors from the integer set $0, \ldots, \lambda$ to the nodes of a graph G in such a way that nodes at a distance of at most two get different colors, while adjacent nodes get colors which are at least two apart. The aim is to minimize λ.

This problem has its roots in mobile computing. The task is to assign radio frequencies to transmitters at different locations without causing interference. This situation can be modeled by a graph, whose nodes are the radio transmitters/receivers, and adjacencies indicate possible communications and, hence, interference. The aim is to minimize the frequency bandwidth, i.e. λ.

In general, both determining the minimum number of necessary colors [10] and deciding if this number is $<k$ for any fixed $k \geq 4$ [9] is NP-complete. Therefore, researchers have focused on some special classes of graphs. For some classes – such as paths, cycles, wheels, tilings and k-partite graphs – tight bounds for the number of colors necessary for an $L(2, 1)$-labeling are well known in the literature and so a coloring can be computed efficiently. For many other classes of graphs – such as chordal graphs [14], interval graphs [8], split graphs [2], outerplanar and planar graphs [2,6], bipartite permutation graphs [1], and co-comparability graphs [5] – approximate bounds have been looked for. For a complete survey, see [4].

Unigraphs [11,12] are graphs uniquely determined by their own degree sequence up to isomorphism and are a superclass including *matrogenic graphs*, *matroidal graphs*, *split matrogenic graphs* and *threshold graphs*. The interested reader can find information related to these classes of graphs in [13].

In [7], all these subclasses are $L(2, 1)$-labeled: threshold graphs can be optimally $L(2, 1)$-labeled in time linear in Δ with $\lambda \leq 2\Delta$, while for matrogenic graphs the upper bound $\lambda \leq 3\Delta$ holds, where Δ is the maximum degree of the graph. In the same paper the problem of $L(2, 1)$-labeling the whole superclass of unigraphs is left open.

In this paper, a $3/2$-approximate algorithm for the $L(2, 1)$-labeling of unigraphs is presented. This algorithm runs in $O(n)$ time, which is the best possible. Observe that a naive algorithm, based on the greedy technique, would obtain an $O(m)$ time complexity, that may be near to $\Theta(n^2)$ for unigraphs.

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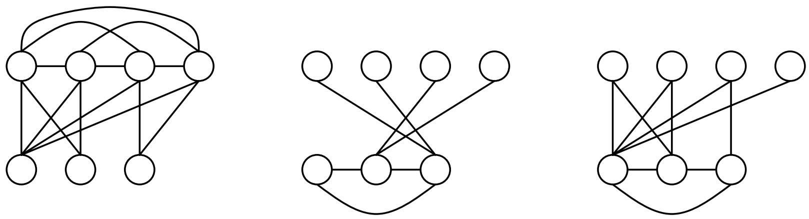

Fig. 1. (a) A split graph G; (b) its complement $\overline{G}$; (c) its inverse G^I.

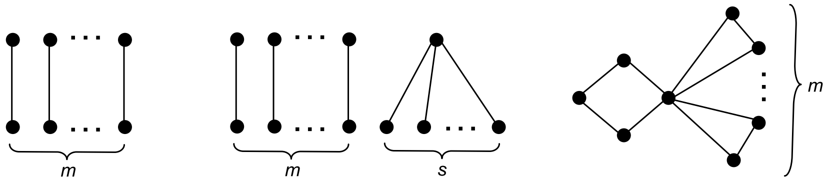

Fig. 2. (a) mK_2; (b) $U_2(m, s)$; (c) $U_3(m)$.

The technique used in the algorithm takes advantage of the degree sequence analysis. In particular, this algorithm exploits the concept of boxes, i.e. the equivalence classes of nodes in a graph under equality of degree.

This paper is organized as follows.

In the next section all the information required for the rest of the paper is summarized. A recognition algorithm for unigraphs and the corresponding characterization theorem on which it is based are outlined in Section 3. The core of the paper comes in the following three sections. Section 4 provides optimal $L(2, 1)$-labeling without repetitions (i.e. $L'(2, 1)$-labeling) for those graphs listed in the characterization of unigraphs, while an $L(2, 1)$-labeling for the same graphs is presented in Section 5. Finally, in Section 6 a linear time (in n and in Δ) 3/2 approximate algorithm for $L(2, 1)$-labeling of unigraphs is presented. Concluding remarks and open problems complete the paper.

2. Preliminaries

In this section all the definitions and known results that will be used in the rest of the paper are summarized.

We consider only finite, simple, loopless graphs $G = (V, E)$, where V and E are the node and edge sets of G with cardinality n and m, respectively. Where no confusion arises, $G = (V, E)$ is called simply G.

Let $DS(G) = \delta_1, \delta_2, \ldots, \delta_n$ be the degree sequence of a graph G sorted by non-increasing values: $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq 0$. We call *boxes* the equivalence classes of nodes in G under equality of degree. In terms of boxes the degree sequence can be compressed as $d_1^{m_1}, d_2^{m_2}, \ldots, d_r^{m_r}$, $d_1 > d_2 > \cdots > d_r \geq 0$, where d_i is the degree of the m_i nodes contained in box $B_i(G)$, $1 \leq m_i \leq n$; hence $\sum_{i=1}^{r} m_i = n$ and $\sum_{i=1}^{r} d_i m_i = 2m$.

We call a box *universal (isolated)* if it contains only universal (isolated) nodes, where a node $x \in V$ is called *universal* (*isolated*) if it is adjacent to all other nodes of V (no other node in V); if x is a universal (isolated) node, then its degree is $d(x) = n - 1$ $(d(x) = 0)$.

A graph I induced by subset $V_I \subseteq V$ is called *complete* or *clique* if any two distinct nodes in V_I are adjacent in G, *stable* or *null* if no two nodes in V_I are adjacent in G.

A graph G is said to be *split* if there is a partition $V = V_K \cup V_S$ of its nodes such that the induced subgraphs K and S are complete and stable, respectively (see Fig. 1(a)).

If $G = (V, E)$ is a graph, its *complement* is $\overline{G} = (V, V \times V - E)$ (see Fig. 1(b)). If $G = (V_K \cup V_S, E)$ is a split graph, its *inverse* G^I is obtained from G by deleting the set of edges $\{\{a_1, a_2\} : a_1, a_2 \in V_K\}$ and adding the set of edges $\{\{b_1, b_2\} : b_1, b_2 \in V_S\}$ (see Fig. 1(c)).

Given a graph G, if its node set V can be partitioned into three disjoint sets V_K, V_S and V_C such that K is a clique, S is a stable set and every node in V_C is adjacent to every node in V_K and to no node in V_S, then the subgraph induced by V_C is called *crown*.

In the following the definitions of some special graphs are recalled [15]:

mK_2: it is the union of m node-disjoint edges $m \geq 1$, also called perfect matching (see Fig. 2(a)).

$U_2(m, s)$: it is the disjoint union of a perfect matching mK_2 and a star $K_{1,s}$, for $m \geq 1$, $s \geq 2$ (see Fig. 2(b)).

$U_3(m)$: for $m \geq 1$, this graph is constructed as follows: fix a node in each component of the graph obtained as disjoint union of the chordless cycle C_4 and m triangles K_3, and merge all these nodes in one (see Fig. 2(c)).

$S_2 = (p_1, q_1; \ldots; p_t, q_t)$: to obtain this graph, add all the edges connecting the centers of l non-isomorphic arbitrary stars K_{1,p_i}, $i = 1, \ldots, t$, each one occurring q_i times, where $p_i, q_i, t \geq 1$, $q_1 + \cdots + q_t = l \geq 2$ (see Fig. 3(a)). Without loss of generality, in the following we assume $p_1 \leq \cdots \leq p_t$.