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On basic forbidden patterns of functions

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ABSTRACT

The allowed patterns of a map on a one-dimensional interval are those permutations that are realized by the relative order of the elements in its orbits. The set of allowed patterns is completely determined by the minimal patterns that are not allowed. These are called basic forbidden patterns.

In this paper, we study basic forbidden patterns of several functions. We show that the logistic map $L_r(x) = rx(1-x)$ and some generalizations have infinitely many of them for $1 < r \leq 4$, and we give a lower bound on the number of basic forbidden patterns of L_4 of each length. Next, we give an upper bound on the length of the shortest forbidden pattern of a piecewise monotone map. Finally, we provide some necessary conditions for a set of permutations to be the set of basic forbidden patterns of such a map.

1. Introduction and definitions

Given a map on a one-dimensional interval, consider the finite sequences (orbits) that are obtained by iterating the map, starting from any point in the interval. The permutations given by the relative order of the elements of these sequences are called *allowed patterns*; permutations that do not appear in this way are called *forbidden patterns*. It was shown in [2,4] that piecewise monotone maps always have forbidden patterns; that is, there are some permutations that do not appear in any orbit. This idea can be used to distinguish random sequences, where every permutation appears with some positive probability, from deterministic sequences produced by iterating a map. Practical aspects of this idea are discussed in [3].

Minimal forbidden patterns, that is, those for which any proper consecutive subpattern is allowed, are called *basic forbidden patterns*. They form an antichain in the partially ordered set of permutations ordered by consecutive pattern containment (see below for definitions), and they contain all the information about the allowed and forbidden patterns of the map.

Consecutive patterns in permutations were first studied in [7] from an enumerative point of view. More recently, they have come up in connection to dynamical systems in [2,4,6].

In this paper, we seek to better understand the set of basic forbidden patterns of functions. Given a map, a natural question is to ask whether its set of basic forbidden patterns is finite or infinite. In Section 2, we give some easy examples of maps with a finite set of basic forbidden patterns. In Section 3, we show that the set of basic forbidden patterns of the logistic map is infinite, and we find some properties of these patterns. We show that the result also holds for a more general class of maps.

Section 4 deals with an important practical question. If we are looking for missing patterns in a sequence in order to tell whether it is random or it has been produced by iterating a piecewise monotone map, it is very useful to have an upper

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bound on the longest patterns whose presence or absence we need to check. In Section 4, we provide an upper bound on the length of the shortest forbidden pattern of a map, based on its number of monotonicity intervals.

Another interesting problem is to characterize what sets of permutations can be the basic forbidden patterns of some piecewise monotone map. In Section 5, we give some necessary conditions that these sets have to satisfy.

1.1. Permutations and consecutive patterns

Denote by $\mathcal{S}_n$ the set of permutations of $\{1, 2, \ldots, n\}$. Let $\mathcal{S} = \cup_{n\geq 1} \mathcal{S}_n$. If $\pi \in \mathcal{S}_n$, we write its one-line notation as $\pi = \pi(1)\pi(2)\cdots\pi(n)$. Sometimes it will be convenient to insert commas between the entries.

Let $x_1, \ldots, x_n \in \mathbb{R}$ with $x_1 < x_2 < \cdots < x_n$. A permutation of $x_1, \ldots, x_n$ can be expressed as $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$, where $\sigma \in \mathcal{S}_n$. We define its *reduction* as

$$\rho(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}) = \sigma(1)\sigma(2)\cdots\sigma(n) = \sigma.$$

In other words, the reduction is a relabeling of the entries with the numbers $1, 2, \ldots, n$ while preserving the order relationships among them. For example, $\rho(3, 4.2, -2, \sqrt{3}, 1) = 45\,132$.

Given two permutations $\pi \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$ with $m \geq n$, we say that π *contains* σ *as a consecutive pattern* if there exists i such that $\rho(\pi(i)\pi(i+1)\cdots\pi(i+n-1)) = \sigma$. In this case, we also say that σ is a consecutive subpattern of π, and we write $\sigma \preceq \pi$. Otherwise, we say that π *avoids* σ *as a consecutive pattern*. In the rest of the paper, all the notions of pattern containment and avoidance refer to the consecutive case, even if the word *consecutive* is omitted. Denote by $\mathrm{Av}_n(\sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid σ as a consecutive pattern, and let $\mathrm{Av}(\sigma) = \cup_{n\geq 1} \mathrm{Av}_n(\sigma)$. In general, if $\Sigma \subset \mathcal{S}$, let $\mathrm{Av}(\Sigma)$ be the set of permutations that avoid all the patterns in Σ, and let $\mathrm{Av}_n(\Sigma) = \mathrm{Av}(\Sigma) \cap \mathcal{S}_n$. Consecutive pattern containment (and avoidance) was first studied in [7]. In [5], the asymptotic behavior of the number of permutations that avoid a consecutive pattern π is considered.

Theorem 1.1 (*[5]*). *Let* $\sigma \in \mathcal{S}_k$ *with* $k \geq 3$. *Then there exist constants* $0 < c, d < 1$ *such that*

$$c^n n! < |\mathrm{Av}_n(\sigma)| < d^n n!$$

for all $n \geq k$.

The consecutive containment relation $\preceq$ defines a partial order on $\mathcal{S}$. Denote by $P_c = (\mathcal{S}, \preceq)$ the resulting infinite partially ordered set. We say that a set $A \subset \mathcal{S}$ is a *closed consecutive permutation class* if it is closed under consecutive pattern containment, that is, if $\pi \in A$ and $\sigma \preceq \pi$ imply that $\sigma \in A$. In this case, the *basis* of A consists of the minimal permutations not in A; that is,

$$\mathrm{Bas}(A) = \{\pi \in \mathcal{S} \setminus A : \text{if } \sigma \preceq \pi, \sigma \neq \pi \text{ then } \sigma \in A\}.$$

Note that $\mathrm{Bas}(A)$ is an antichain in P_c; that is, there are no two permutations $\tau, \pi \in \mathrm{Bas}(A)$ with $\tau \neq \pi$ and $\tau \preceq \pi$. Conversely, any antichain Σ is the basis of the closed class $\mathrm{Av}(\Sigma)$. This gives a one-to-one correspondence between antichains of P_c and closed consecutive permutation classes.

For example, if A is the set of up–down or down–up permutations, i.e., those permutations satisfying $\pi(1) < \pi(2) > \pi(3) < \pi(4) > \cdots$ or $\pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots$, then $\mathrm{Bas}(A) = \{123, 321\}$. If B is the antichain $\{132, 231\}$, then $\mathrm{Av}(B)$ is the set of permutations having no *peaks*, i.e., no i such that $\pi(i-1) < \pi(i) > \pi(i+1)$.

1.2. Allowed and forbidden patterns of maps

Let $f : I \to I$, where $I \subset \mathbb{R}$ is a closed interval. Given $x \in I$ and $n \geq 1$, let

$$\mathrm{Pat}(x, f, n) = \rho(x, f(x), f^2(x), \ldots, f^{n-1}(x)),$$

provided that there is no pair $0 \leq i < j \leq n-1$ such that $f^i(x) \neq f^j(x)$. If such a pair exists, then $\mathrm{Pat}(x, f, n)$ is not defined. When it is defined, $\mathrm{Pat}(x, f, n) \in \mathcal{S}_n$. For example, if $L_4 : [0, 1] \to [0, 1]$ is the *logistic map* $L_4(x) = 4x(1-x)$ and we take $x = 0.8$ to be the initial value, then

$$(x, L_4(x), L_4^2(x), L_4^3(x)) = (0.8, 0.64, 0.9216, 0.28901376),$$

so $\mathrm{Pat}(0.8, L_4, 4) = 3241$.

If $\pi \in \mathcal{S}_n$ and there is some $x \in I$ such that $\mathrm{Pat}(x, f, n) = \pi$, we say that π is *realized* by f (at x), or that π is an *allowed pattern* of f. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f) = \bigcup_{n\geq 1} \mathrm{Allow}_n(f)$, where

$$\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n.$$

The remaining permutations are called *forbidden patterns*, and are denoted by $\mathrm{Forb}(f) = \mathcal{S} \setminus \mathrm{Allow}(f)$.

It is noticed in [6] that $\mathrm{Allow}(f)$ is closed under consecutive pattern containment: if $\mathrm{Pat}(x, f, n) = \pi$ and $\tau \preceq \pi$, then there exist i, j such that $\rho(\pi(i)\pi(i+1)\cdots\pi(j)) = \tau$; hence $\mathrm{Pat}(f^{i-1}(x), f, j-i+1) = \tau$. Those forbidden patterns for which any proper subpattern is allowed are called the *basic forbidden patterns* of f, and are denoted $\mathrm{B}(f)$. This set is an antichain