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Characterization and representation problems for intersection betweennesses

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ABSTRACT

For a set system $\mathcal{M} = (M_v)_{v\in V}$ indexed by the elements of a finite set V, the intersection betweenness $\mathcal{B}(\mathcal{M})$ induced by $\mathcal{M}$ consists of all triples $(u, v, w) \in V^3$ with $M_u \cap M_w \subseteq M_v$. Similarly, the strict intersection betweenness $\mathcal{B}_s(\mathcal{M})$ induced by $\mathcal{M}$ consists of all triples $(u, v, w) \in \mathcal{B}(\mathcal{M})$ such that u, v, and w are pairwise distinct. The notion of a strict intersection betweenness was introduced by Burigana [L. Burigana, Tree representations of betweenness relations defined by intersection and inclusion, Math. Soc. Sci. 185 (2009) 5–36]. We provide axiomatic characterizations of intersection betweennesses and strict intersection betweennesses. Our results yield a simple and efficient algorithm that constructs a representing set system for a given (strict) intersection betweenness. We study graphs whose strict shortest path betweenness is a strict intersection betweenness. Finally, we explain how the algorithmic problem related to Burigana's notion of a partial tree representation can be solved efficiently using well-known algorithms.

1. Introduction

In this paper, we study so-called betweennesses induced by graphs as well as set systems. Betweennesses capture and generalize in an abstract way natural geometric properties of points in $\mathbb{R}^n$, and the axiomatic study of betweenness as a mathematical concept goes back to Huntington and Kline [13] in 1917. Algorithmic problems related to betweennesses have been studied as relaxations of ordinal embeddings [1,14,12] and occur for instance in psychometrics [3] and as arrangement problems in molecular biology [6,10]. For such betweenness problems, several strong hardness results have been obtained [5,18,4] and only a few positive results are known [5,12,15].

We consider finite, simple, and undirected graphs as well as finite set systems defined over finite ground sets. Let G be a graph. The vertex set of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. For a vertex u of G, the neighbourhood $N_G(u)$ in G equals $\{v \in V(G) \mid uv \in E(G)\}$ and the degree $d_G(u)$ in G equals $|N_G(u)|$. A path P of length l in G between two vertices v_0 and v_l of G is a sequence $P{:}v_0v_1 \dots v_l$ of $l + 1$ distinct vertices $v_0, v_1, \dots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for $1 \le i \le l$. The distance $\mathrm{dist}_G(u, v)$ in G between two vertices u and v of G is the minimum length of a path in G between u and v. A cycle C of length l in G is a sequence $C{:}v_1v_2 \dots v_lv_1$ such that $v_1v_2 \dots v_l$ is a path in G and $v_1v_l \in E(G)$. For a finite set V, V^3 denotes the set of all ordered triples of elements of V. A triple $(u, v, w) \in V^3$ is called strict if u, v, and w are

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pairwise distinct. Let V_s^3 denote the set of all strict triples in V^3. For $k \in \mathbb{N}_0$, let $\binom{V}{k}$ denote the set of all subsets of V that are of cardinality k.

For a graph G, the *shortest path betweenness* $\mathcal{B}(G)$ *of* G consists of all triples $(u, v, w) \in V(G)^3$ such that v lies on a shortest path in G between u and w, or equivalently $(u, v, w) \in V(G)^3$ belongs to $\mathcal{B}(G)$ if and only if $\mathrm{dist}_G(u, w) = \mathrm{dist}_G(u, v) + \mathrm{dist}_G(v, w) < \infty$. The *strict shortest path betweenness* $\mathcal{B}_s(G)$ *of* G consists of all strict triples in $\mathcal{B}(G)$, i.e. $\mathcal{B}_s(G) = \mathcal{B}(G) \cap V(G)_s^3$. Shortest path betweennesses are a special case of betweennesses induced by metrics, which were first studied by Menger in 1928 [17]. The shortest path betweennesses of trees have received special attention, and several different axiomatic characterizations have been proposed [3,7,19,20]. For a tree/forest T, we call $\mathcal{B}(T)$ the *tree/forest betweenness of* T and $\mathcal{B}_s(T)$ the *strict tree/forest betweenness of* T.

In [3], Burigana introduces a betweenness notion derived from set systems. For a set system $\mathcal{M} = (M_v)_{v \in V}$ indexed by the elements of a finite set V, the *intersection betweenness* $\mathcal{B}(\mathcal{M})$ *induced by* $\mathcal{M}$ consists of all triples $(u, v, w) \in V^3$ with $M_u \cap M_w \subseteq M_v$, and the *strict intersection betweenness* $\mathcal{B}_s(\mathcal{M})$ *induced by* $\mathcal{M}$ consists of all strict triples $(u, v, w) \in V_s^3$ with $M_u \cap M_w \subseteq M_v$, i.e.

$$\mathcal{B}(\mathcal{M}) = \{(u, v, w) \in V^3 \mid M_u \cap M_w \subseteq M_v\}$$
$$\mathcal{B}_s(\mathcal{M}) = \mathcal{B}(\mathcal{M}) \cap V_s^3 = \{(u, v, w) \in V_s^3 \mid M_u \cap M_w \subseteq M_v\}.$$

For every finite set V, a set $\mathcal{B} \subseteq V^3$ is an *intersection betweenness* if there is some set system $\mathcal{M}$ with $\mathcal{B} = \mathcal{B}(\mathcal{M})$. Similarly, a set $\mathcal{B} \subseteq V_s^3$ is a *strict intersection betweenness* if there is some set system $\mathcal{M}$ with $\mathcal{B} = \mathcal{B}_s(\mathcal{M})$.

Burigana provides some axioms for strict intersection betweennesses. Furthermore, he characterizes the two classes of strict intersection betweennesses that coincide with some strict tree betweenness (betweennesses with a *full tree representation*), and that contain some strict tree betweenness (betweennesses with a *partial tree representation*). A central problem left open in [3] is the (axiomatic) characterization of (strict) intersection betweennesses. Furthermore, the procedure proposed in [3] for the solution of the partial tree representation problem does not lead to an efficient algorithm.

Our results are as follows. In Section 2, we provide axiomatic characterizations of intersection betweennesses and strict intersection betweennesses thus solving the problem left open in [3]. Furthermore, our results yield a simple and efficient algorithm that constructs a representing set system for a given (strict) intersection betweenness. In Section 3, we characterize those graphs whose strict shortest path betweenness is a strict intersection betweenness. Furthermore, we describe representations of strict tree betweennesses as strict intersection betweennesses of set systems over small ground sets. Finally, in Section 4, we explain how the algorithmic problem related to Burigana's partial tree representation can be solved efficiently using well-known algorithms.

2. Characterizing and representing intersection betweennesses

Burigana [3] provides the following three axioms, which he claims to hold for every strict intersection betweenness $\mathcal{B}$.

(I_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B}$.
(I_2) $\forall u, v, w, z \in V : (u, v, w), (u, z, v) \in \mathcal{B} \Rightarrow (u, z, w) \in \mathcal{B}$.
(I_3) $\forall u, v, w, t, z \in V : (t, u, z), (t, w, z), (u, v, w) \in \mathcal{B} \Rightarrow (t, v, z) \in \mathcal{B}$.

These three axioms clearly hold for every intersection betweenness because of elementary properties of set intersection and inclusion. Furthermore, (I_1) also holds for every strict intersection betweenness. Contrary to Burigana's claim, the properties (I_2) and (I_3) are actually problematic for strict intersection betweennesses because they potentially imply the presence of non-strict triples. In order to ensure that the triple (u, z, w), whose existence is guaranteed by (I_2), is strict, one has to add the condition $w \neq z$. Similarly, in order to ensure that the triple (t, v, z), whose existence is guaranteed by (I_3), is strict, one has to add the condition $t \neq v \neq z$. This leads to the following modified versions of (I_2) and (I_3).

(I_2^s) $\forall u, v, w, z \in V : (u, v, w), (u, z, v) \in \mathcal{B}$ and $w \neq z \Rightarrow (u, z, w) \in \mathcal{B}$.
(I_3^s) $\forall u, v, w, t, z \in V : (t, u, z), (t, w, z), (u, v, w) \in \mathcal{B}$ and $t \neq v \neq z \Rightarrow (t, v, z) \in \mathcal{B}$.

While strict intersection betweennesses can be empty, intersection betweennesses always contain all triples of the form (u, u, w) and (u, w, w). Therefore, they necessarily satisfy another property.

(I_4) $\forall u, w \in V : (u, u, w) \in \mathcal{B}$.

Note that (I_4) together with (I_3) actually implies (I_2) by choosing $t = u$ in (I_3).

We now show that the above axioms yield characterizations of intersection betweennesses and strict intersection betweennesses. Furthermore, we also prove that there are always representing set systems over quadratic ground sets, which can be constructed efficiently.

Theorem 1. *Let V be a finite set and let $\mathcal{B} \subseteq V^3$.*

(i) *If there is a set system $\mathcal{M} = (M_v)_{v \in V}$ with $\mathcal{B} = \mathcal{B}(\mathcal{M})$, then $\mathcal{B}$ satisfies* (I_1), (I_3), *and* (I_4).