Discrete Applied Mathematics 158 (2010) 2075–2080

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Facet-inducing web and antiweb inequalities for the graph coloring polytope

Gintaras Palubeckis *

Faculty of Informatics, Kaunas University of Technology, Studentu 50-408, 51368 Kaunas, Lithuania

ARTICLE INFO

Article history:
Received 26 October 2009
Received in revised form 3 August 2010
Accepted 10 August 2010
Available online 30 August 2010

Keywords:
Polyhedral combinatorics
Graph coloring
Polytopes

ABSTRACT

For a graph G and its complement $\bar{G}$, we define the graph coloring polytope $P(G)$ to be the convex hull of the incidence vectors of star partitions of $\bar{G}$. We examine inequalities whose support graphs are webs and antiwebs appearing as induced subgraphs in G. We show that for an antiweb $\bar{W}$ in G the corresponding inequality is facet-inducing for $P(G)$ if and only if $\bar{W}$ is critical with respect to vertex colorings. An analogous result is also proved for the web inequalities.

1. Introduction

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices share the same color. Let $\bar{G} = (V, \bar{E})$ denote the complement of the graph $G = (V, E)$. We relate to cliques in $\bar{G}$ the graphs called stars. A *star* is a tree S with vertex set $\{v_1, \dots, v_t\}$, $t \geqslant 1$, and, provided $t > 1$, with the nonempty edge set $\{(v_1, v_i) \mid i = 2, \dots, t\}$. We denote by $c(S) = v_1$ the *center vertex* of the star S. By a *star partition* of $\bar{G}$ we will understand a collection Π of stars $S_i = (V_i, E_i)$, $i = 1, \dots, l$, such that $V_i \cap V_j = \emptyset$ for each pair i, j, $i \neq j$, $\cup_{i=1}^{l} V_i = V$, and V_i for each $i \in \{1, \dots, l\}$ is a clique in $\bar{G}$. To a coloring of G, we can associate a star partition of $\bar{G}$. In this paper, we will assume that V is a set of integers treated as unique identifiers assigned to the vertices of G. We say that a star partition Π is *admissible* if, for each star $S_i = (V_i, E_i)$ in Π, $c(S_i) = \min_{j \in V_i} j$. There is a one-to-one correspondence between colorings of G and admissible star partitions of $\bar{G}$. In [16], we used this relation to define the graph coloring polytope $P(G) = \mathrm{conv}\{x(\Pi) \mid \Pi \in \Psi(\bar{G})\}$, where $\Psi(\bar{G})$ is the set of all admissible star partitions of $\bar{G}$ and $x(\Pi) = (x_{ij}(\Pi) \mid (i, j) \in \bar{E})$ is the incidence vector of Π, i.e., 0–1 vector with $x_{ij}(\Pi) = 1$ if and only if $(i, j) \in E_k$ for some star S_k in Π.

Several different graph coloring polytopes have been studied by other authors. Coll et al. [5] and Méndez Díaz and Zabala [13] considered polytopes in the space of n^2 binary variables representing all combinations of vertices of an n-vertex graph and n possible colors to color them as well as n extra variables needed to represent colors. Figueiredo et al. [10] introduced the graph coloring polytope associated with a model relating acyclic orientations of a graph to its chromatic number. Campêlo et al. [3] studied the polytope arising from the asymmetric representatives formulation of the graph coloring problem. Cornaz [6,7] investigated the polytope defined as the convex hull of the incidence vectors of the clique-connecting forests of G. Optimizing a certain linear function over the clique-connecting forest polytope is equivalent to determining the chromatic number of the graph. In [7], Cornaz derived a class of nontrivial facet-inducing inequalities for this polytope. Recently, Hansen et al. [12] presented polyhedral results for the graph coloring problem, which were obtained using general set packing and set covering polytopes.

* Fax: +370 37 300352.
E-mail address: gintaras@soften.ktu.lt.

0166-218X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2010.08.016

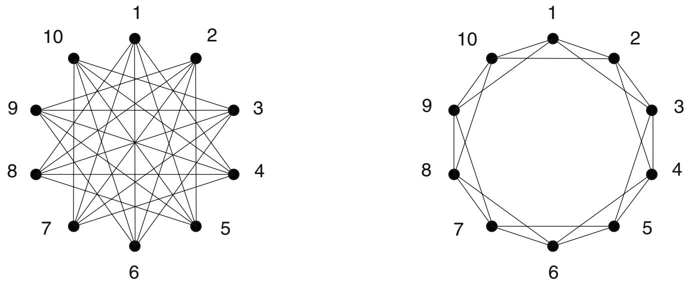

Fig. 1. Left: web W_{10}^{3}; right: antiweb $\bar{W}_{10}^{3}$.

As is already known [16], our definition of a graph coloring polytope allows us to easily uncover the inherent relationship between graph coloring and finding a maximum independent (stable) set in a graph. To see this, let $T = \{(i, j, k) \mid i < \min\{j, k\}$ and $\{i, j, k\}$ forms a triangle in $\bar{G}\}$, $Z = \{(i, j, k) \mid (i, j), (j, k) \in \bar{E}, (i, k) \notin \bar{E}\}$. For a star partition $\Pi \in \Psi(\bar{G})$, its incidence vector $x = x(\Pi)$ satisfies the following inequalities

$$x_{ij} + x_{jk} \leqslant 1 \quad \text{for all } (i, j, k) \in T \cup Z. \tag{1}$$

Each binary vector satisfying (1) defines an independent set in the graph $H_G = (V(H), E(H))$ with vertices $v_{ij} \in V(H)$ corresponding to $(i, j) \in \bar{E}$ and edges $(v_{ij}, v_{jk}) \in E(H)$ corresponding to $(i, j), (j, k) \in \bar{E}$ such that $(i, j, k) \in T \cup Z$. Essentially the same reduction from graph coloring to the maximum stable set problem was independently discovered by Cornaz and Jost [6,8]. They consider the relationship between colorings of G and stellar forests of $\bar{G}$ and construct a graph that coincides with the graph H_G defined above. Let $\alpha(G)$ denote the stability number of a graph G. In [16] and independently in [6,8], the following result is shown.

Theorem 1. *For a simple n-vertex graph G,* $\chi(G) + \alpha(H_G) = n$.

Thus the graph G can be optimally colored by way of finding a maximum independent set in the graph H_G. The latter problem belongs to the famous class of quadratic pseudo-Boolean optimization problems [2].

Using (1), the polytope $P(G)$ can be written as $P(G) = \operatorname{conv}\{x = (x_{ij}), (i, j) \in \bar{E} \mid x \text{ is binary and satisfies (1)}\}$. Let $P_{\text{stable}}(G)$ denote the *stable set polytope* of a graph G, that is, the convex hull of the incidence vectors of the independent (stable) sets of G. From the above reformulation of $P(G)$, it follows that $P(G) = P_{\text{stable}}(H_G)$. Thus, a set of valid and facet-inducing inequalities for $P_{\text{stable}}(H_G)$ provides a (partial) linear description of the polytope $P(G)$. In the literature, such inequalities for $P_{\text{stable}}(G)$ typically are defined in terms of subgraphs of G that are isomorphic to quite simple well-structured graphs like complete graphs, odd holes, odd antiholes, wheels, webs and antiwebs. In the case of H_G, the vertices of such subgraphs correspond to edges of $\bar{G}$. However, of far greater interest is that the subgraphs producing valid inequalities for $P(G)$ are to be taken from G (or $\bar{G}$), and not from H_G. Several classes of facet-inducing inequalities for $P(G)$ were presented in [17]. These inequalities are derived from independent sets, odd holes, odd wheels, and odd antiholes in $\bar{G}$. The relation between graph coloring and stable set polytopes was also investigated in [6,7]. In particular, in [7] it is shown that the nontrivial and nondegenerate facets of the stable set polytope are facets of the clique-connecting forest polytope.

In this note, the focus is on webs and antiwebs considered as subgraphs of G. Both webs and antiwebs received a considerable amount of attention in the literature dealing with the stable set polytopes. Some recent papers on this topic include [1,4,9,11,14,15,18–20]. In the present work, we show that, under certain conditions, antiwebs (Section 2) and webs (Section 3) in G give rise to facets of the graph coloring polytope $P(G)$.

We conclude the introduction with a few notations. For a graph G and its vertex u, we denote by $N_u(G)$ the neighborhood of u. We let $S(U)$ stand for the star corresponding to the clique $U \subset V$. The center vertex of $S(U)$ is uniquely determined: it is the vertex $u \in U$ such that $u < v$ for each $v \in U \setminus \{u\}$. Given a star partition Π, we will write $E(\Pi)$ for the union of the edge sets of all stars in Π. For an inequality $\sum_{(i,j)\in\bar{E}} b_{ij}x_{ij} \leqslant b_0$, we will assume that the order of the subscripts of b does not matter, i.e., b_{ij} and b_{ji} refer to the same coefficient.

2. Antiweb inequalities

Let m and p be integers satisfying $p \geqslant 2$ and $m \geqslant 2p + 1$. As defined by Trotter [21], the web $W_m^p = (V(W), E(W))$ is a graph with the vertex set $V(W) = \{v_1, \ldots, v_m\}$ and the edge set $E(W) = \{(v_i, v_j) \mid v_i, v_j \in V(W) \text{ and } p \leqslant |i - j| \leqslant m - p\}$. An example of the web is shown in Fig. 1, where $v_i = i, i = 1, \ldots, m$. The antiweb $\bar{W}_m^p = (V(W), E(\bar{W}))$ is the complement of the web W_m^p (see Fig. 1).

Before addressing antiweb-based facets of $P(G)$, we state the following simple fact.