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The induced path function, monotonicity and betweenness

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ABSTRACT

The geodesic interval function I of a connected graph allows an axiomatic characterization involving axioms on the function only, without any reference to distance, as was shown by Nebeský [20]. Surprisingly, Nebeský [23] showed that, if no further restrictions are imposed, the induced path function J of a connected graph G does not allow such an axiomatic characterization. Here $J(u, v)$ consists of the set of vertices lying on the induced paths between u and v. This function is a special instance of a transit function. In this paper we address the question what kind of restrictions could be imposed to obtain axiomatic characterizations of J. The function J satisfies betweenness if $w \in J(u, v)$, with $w \neq u$, implies $u \notin J(w, v)$ and $x \in J(u, v)$ implies $J(u, x) \subseteq J(u, v)$. It is monotone if $x, y \in J(u, v)$ implies $J(x, y) \subseteq J(u, v)$. In the case where we restrict ourselves to functions J that satisfy betweenness, or monotonicity, we are able to provide such axiomatic characterizations of J by transit axioms only. The graphs involved can all be characterized by forbidden subgraphs.

1. Introduction

In [18] the notion of transit function is introduced as a means to study how to move around in discrete structures. Basically, it is a function satisfying three simple axioms on a set V, which is provided with a structure σ. Prime examples of such structures are: a set of edges E, so that we are considering a graph $G = (V, E)$, or a partial ordering $\leq$, so that we are considering a partially ordered set $(V, \leq)$. The idea is to study transit functions that have additional properties defined in terms of the structure σ. For instance, the transit function may be defined in terms of paths in the graph $G = (V, E)$. Such transit functions are called *path transit functions* on G in [18]. A prime example is the geodesic interval function $I : V \times V \to 2^V$ of a connected graph G, where $I(u, v)$ is the set of vertices lying on the shortest paths between u and v. This function has been widely studied from many different perspectives, to name a few: convexity, see e.g. [10,17,29], medians, see e.g. [14,17], monotonicity, see e.g. [15,17,24]. For the induced path function $J : V \times V \to 2^V$ of a connected graph G, where $J(u, v)$ is the set of vertices lying on the induced paths between u and v, similar questions and problems have been studied: convexity, see e.g. [4,9,11,13,16], median-type properties, see [16], monotonicity, see e.g. [3–5]. This exemplifies the basic idea for introducing the concept of transit function in [18]: transfer ideas, questions and problems from one transit function to another and see whether interesting problems arise. This was the motivation to study the analogues of these questions for the all-paths function A on a graph: now $A(u, v)$ consists of the vertices on the u, v-paths, see [2]. The convexity related to the all-paths function was already studied much earlier, see e.g. [8,26]. Note that any transit function has an associated convexity. Such convexities are called interval convexities in [1,29]. Those related to path transit functions are discussed in more detail in [6].

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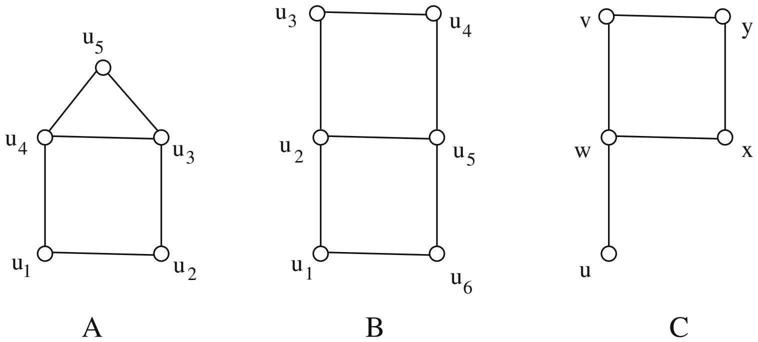

Fig. 1. A: house, B: domino, C: P-graph.

In [20–22] Nebeský obtained some quite interesting results, see also [19]. He characterized the functions that are the geodesic interval function of some graph without any reference to the notion of distance. That is, a function $I : V \times V \to 2^V$ is the geodesic interval function of some connected graph if and only if I satisfies a set of axioms that are phrased in terms of I only. This immediately poses the problem for other transit functions on graphs: can they be characterized in terms of such transit axioms only? For the all-paths function A this was done in [2]. Surprisingly, such a characterization of the induced path function J is *not* possible, as was shown by Nebeský in [23] using first order logic. This poses the problem whether it is still possible to characterize the induced path function if some further restrictions are imposed, or if the graph satisfies some extra properties.

The *aim of this paper* is to study special cases, in which J can indeed be characterized by transit axioms only. Then one searches for the appropriate properties of the graphs and the appropriate transit axioms for J. These cases are where J has the properties of a betweenness, and where J is monotone, that is, all sets $J(u, v)$ are J-convex. As one might expect, the characterizations we seek for J in this paper involve forbidden (induced) subgraphs for the graphs. The most important ones are the house, the domino and the P-graph, see Fig. 1, and the holes. Here a hole, or a long cycle, is a cycle with at least 5 vertices. The so-called *HHD*-free graphs and *HHP*-free graphs that appear over and over below also have other interesting aspects. Here *H* stands for house or hole, *D* for domino, and *P* for P-graph. These classes of graphs have important applications as far as elimination orderings in graphs are concerned. *HHD*-free and *HHP*-free graphs are natural generalizations of the class of chordal graphs in connection with the lexicographic breadth first search (LexBFS) and maximum cardinality search (MCS) orderings in graphs, see [25,28]. In [7], using a relaxation of the induced path convexity known as m^3-convexity, it is proved that graphs, for which LexBFS (MCS) is a semi-simplicial ordering, constitute precisely the class of *HHD*-free (*HHP*-free) graphs. See also [12].

The paper is organized as follows. In Section 2 we give the definition of transit function, betweenness and monotonicity, and introduce five new axioms for the characterization of the *induced path function J* in terms of these transit axioms. Each of these new axioms captures some aspect of the idea of betweenness that is exemplified in the geodesic interval function. Moreover we prove some first results involving J and betweenness and monotonicity. In Section 4 we prove our main results, viz. Theorems 2 and 3: a transit function that is a betweenness and satisfies in addition some of the five new axioms necessarily is the induced path function of some connected graph. Using the above characterizations, we also characterize the classes of *HHD*-free and *HHP*-free graphs by the induced path function.

2. Transit functions and betweenness

In this section we collect the necessary terminology on transit functions and betweenness and establish some first results. A graph is said to be *HHD*-free if it does not contain a house, a hole or a domino as an induced subgraph. It is called *HHP*-free if it does not contain a house, a hole or a P-graph as induced subgraph. A hole is a cycle of length at least 5, for the other graphs see Fig. 1.

Let V be a finite set. A *transit function* on V is a function $R : V \times V :\to 2^V$ satisfying the following three axioms:

(t1) $u \in R(u, v)$, for any u and v in V,
(t2) $R(u, v) = R(v, u)$, for all u and v in V,
(t3) $R(u, u) = \{u\}$, for all u in V.

A subset W of V is *R-convex* if $R(u, v) \subseteq W$, for any two vertices u, v in W. If, moreover, $G = (V, E)$ is a graph with vertex set V, then we say that R is a *transit function on G*. Note that the above axioms do not reflect any aspect of the graph G. But our interest will be in transit functions that are defined in terms of the graph. Then the challenge is whether these graphical properties of the transit function can be characterized by transit axioms that are in terms of the transit function only.

The *underlying graph* G_R of a transit function R is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u, v) = \{u, v\}$. Note that, in general, G and G_R will not be isomorphic graphs. Transit