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Equality in a linear Vizing-like relation that relates the size and total domination number of a graph

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ABSTRACT

Let G be a graph, each component of which has order at least 3, and let G have order n, size m, total domination number γ_t and maximum degree $\Delta(G)$. Let $\Delta = 3$ if $\Delta(G) = 2$ and $\Delta = \Delta(G)$ if $\Delta(G) \geq 3$. It is known [M.A. Henning, A linear Vizing-like relation relating the size and total domination number of a graph, J. Graph Theory 49 (2005) 285–290; E. Shan, L. Kang, M.A. Henning, Erratum to: a linear Vizing-like relation relating the size and total domination number of a graph, J. Graph Theory 54 (2007) 350–353] that $m \leq \Delta(n - \gamma_t)$. In this paper we characterize the extremal graphs G satisfying $m = \Delta(n - \gamma_t)$.

1. Introduction

In this paper we continue the study of total domination in graphs. Let $G = (V, E)$ be a graph with vertex set V, edge set E and no isolated vertex. A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination in graphs is now well studied in graph theory. The literature on the subject has been surveyed and detailed in the recent book [12]. A survey of total domination in graphs can also be found in [9].

A classical result of Vizing [18] relates the size and the ordinary domination number, γ, of a graph of given order. Rautenbach [14] shows that the square dependence on n and γ in the result of Vizing turns into a linear dependence on n, γ, and the maximum degree Δ.

Dankelmann et al. [4] proved a Vizing-like relation between the size and the total domination number of a graph of given order. Sanchis [15] showed that if we restrict our attention to connected graphs with total domination number at least 5, then the bound in [4] can be improved slightly. The square dependence on n and γ_t presented in [4,15] is improved in [8,16,19] into a linear dependence on n, γ_t and Δ by demanding a more even distribution of the edges by restricting the maximum degree Δ. In particular, the following linear Vizing-like relation relating the size of a graph and its order, total domination number, and maximum degree is established in [8,16].

Theorem A ([8,16]). *Let G be a graph each component of which has order at least 3, and let G have order n, size m, total domination number γ_t, and maximum degree $\Delta(G)$. Let $\Delta = 3$ if $\Delta(G) = 2$ and $\Delta = \Delta(G)$ if $\Delta(G) \geq 3$. Then, $m \leq \Delta(n - \gamma_t)$.*

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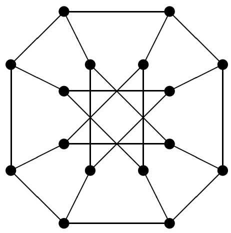

Fig. 1. The generalized Petersen graph GP_{16} of order 16.

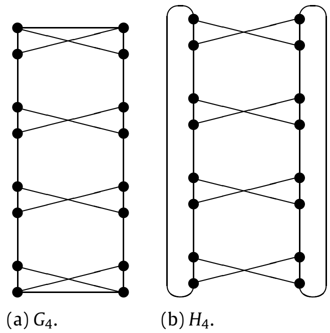

Fig. 2. Cubic graphs $G_4 \in \mathcal{G}$ and $H_4 \in \mathcal{H}$.

Our aim in this paper is to characterize the extremal graphs achieving equality in the upper bound in Theorem A; that is, to characterize the graphs G satisfying the statement of Theorem A such that $m = \Delta(n - \gamma_t)$.

1.1. Notation

For notation and graph theory terminology we in general follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n(G) = |V|$ and edge set E of size $m(G) = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $d_G(v) = |N_G(v)|$. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex adjacent to a vertex of degree 1 is called a *support* vertex. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \cup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we simply write $N(v)$ and $d(v)$ rather than $N_G(v)$ and $d_G(v)$, respectively. Further we write $N[v]$, $N[S]$ and $N(S)$ rather that $N_G[v]$, $N_G[S]$ and $N_G(S)$, respectively. For sets $A, B \subseteq V$, we say that A *dominates* B if $B \subseteq N[A]$, while A *totally dominates* B if $B \subseteq N(A)$.

For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. Further if $S \neq V$, then we denote the graph obtained from G by deleting all vertices in S by $G - S$. A component of G that is isomorphic to a graph F is called an F-component of G.

A *cycle* on n vertices is denoted by C_n, while a *path* on n vertices is denoted by P_n. We denote by K_n the *complete graph* on n vertices. A 2-*path* in G is a path on at least three vertices with both ends of the path having degree at least 3 in G and with every internal vertex of the path having degree 2 in G. A *special* 2-*path* in G is a 2-path $v_1v_2v_3v_4v_5$ such that v_1 and v_5 have two common neighbors, x and y say, in G, and the vertices v_1, v_5, x and y all have degree 3 in G. In particular, we note that $N(v_1) = \{v_2, x, y\}$ and $N(v_5) = \{v_4, x, y\}$.

2. Special graphs and families of graphs

2.1. The family $\mathcal{G}_{\text{cubic}}$

Let GP_{16} denote the generalized Petersen graph of order 16 shown in Fig. 1.

The following two infinite families $\mathcal{G}$ and $\mathcal{H}$ of connected cubic graphs (described below) with total domination number one-half their orders are constructed in [5]. For $k \geq 1$, let G_k be the graph constructed as follows. Consider two copies of the path P_{2k} with respective vertex sequences $a_1b_1a_2b_2 \dots a_kb_k$ and $c_1d_1c_2d_2 \dots c_kd_k$. Let $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$, $C = \{c_1, c_2, \dots, c_k\}$, and $D = \{d_1, d_2, \dots, d_k\}$. For each $i \in \{1, 2, \dots, k\}$, join a_i to d_i and b_i to c_i. To complete the construction of the graph $G_k \in \mathcal{G}$ join a_1 to c_1 and b_k to d_k. Let $\mathcal{G} = \{G_k \mid k \geq 1\}$. For $k \geq 2$, let H_k be obtained from G_k by deleting the two edges a_1c_1 and b_kd_k and adding the two edges a_1b_k and c_1d_k. Let $\mathcal{H} = \{H_k \mid k \geq 2\}$. We note that G_k and H_k are cubic graphs of order $4k$. Further, we note that $G_1 = K_4$. The graphs $G_4 \in \mathcal{G}$ and $H_4 \in \mathcal{H}$, for example, are illustrated in Fig. 2.

Let $\mathcal{G}_{\text{cubic}} = \mathcal{G} \cup \mathcal{H} \cup \{GP_{16}\}$. We note that each graph in the family $\mathcal{G}_{\text{cubic}}$ is a cubic graph.