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A combinatorial approach to a general two-term recurrence

Toufik Mansour [a,*], Mark Shattuck [b]

[a] Department of Mathematics, University of Haifa, 31905 Haifa, Israel
[b] Department of Mathematics, University of Tennessee, 37996 Knoxville, TN, USA

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ABSTRACT

We provide combinatorial proofs of explicit formulas for some sequences satisfying particular cases of the general recurrence $\left|{n \atop k}\right| = (\alpha(n-1)+\beta k+\gamma)\left|{n-1 \atop k}\right| + (\alpha'(n-1)+\beta' k+\gamma')\left|{n-1 \atop k-1}\right| + [n=k=0]$, which have been previously shown using other methods. Many interesting combinatorial sequences are special cases of this recurrence, such as binomial coefficients, both kinds of Stirling numbers, Lah numbers, and two types of Eulerian numbers. Among the cases we consider are $\alpha' = 0$, $\alpha = -\beta$, and $\beta = \beta' = 0$. We also provide combinatorial proofs of some prior identities satisfied by $\left|{n \atop k}\right|$ when $\alpha' = 0$ and when $\beta = \beta' = 0$ as well as deduce some new ones in the former case. In addition, we introduce a polynomial generalization of $\left|{n \atop k}\right|$ when $\alpha' = 0$ which has among its special cases q-analogues of both kinds of Stirling numbers. Finally, we supply combinatorial proofs of two formulas relating binomial coefficients and the two kinds of Stirling numbers which were previously obtained by equating three different expressions for the solution of the aforementioned recurrence in the case when $\alpha' = \beta' = 0$ and all other weights are unity.

1. Introduction

Graham, Knuth, and Patashnik propose the following open problem in their text *Concrete Mathematics* [6, p. 319, Problem 6.94]:

> Develop a general theory of the solutions to the two-parameter recurrence
>
> $$\left|{n \atop k}\right| = (\alpha n+\beta k+\gamma)\left|{n-1 \atop k}\right| + (\alpha' n+\beta' k+\gamma')\left|{n-1 \atop k-1}\right| + [n=k=0], \quad \text{for } n,k \geq 0, \tag{1}$$
>
> assuming that $\left|{n \atop k}\right| = 0$ when $n < 0$ or $k < 0$. What special values $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ yield "fundamental solutions" in terms of which the general solution can be expressed?

Many combinatorial sequences of interest satisfy recurrences that are special cases of (1), which include binomial coefficients (see A007318 in [15]), both kinds of Stirling numbers (A008275, A008277), Lah numbers (A008297), two types of Eulerian numbers (A008292, A008517), and two types of associated Stirling numbers (A008306, A008299). Furthermore, several of the generalizations of the Stirling numbers that have been studied also satisfy recurrences of the form (1); see, for example, [7–9].

* Corresponding author. Tel.: +972 4 8240705; fax: +972 4 8240024.
E-mail addresses: toufik@math.haifa.ac.il, tmansour@univ.haifa.ac.il (T. Mansour), shattuck@math.utk.edu (M. Shattuck).

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Perhaps the most general result so far concerning solutions to (1) is due to Neuwirth [12], who shows that if $\alpha' = 0$, then

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = \prod_{i=1}^{k}(\beta' i + \gamma') \sum_{i=0}^{n}\sum_{j=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \binom{i}{j} \begin{Bmatrix} j \\ k \end{Bmatrix} \alpha^{n-i}\beta^{j-k}(\gamma+\alpha)^{i-j}, \quad n, k \geq 0, \tag{2}$$

where $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]$ and $\left\{\begin{smallmatrix} m \\ r \end{smallmatrix}\right\}$ denote the Stirling numbers of the first and second kind, respectively. In deriving Eq. (2), Neuwirth uses infinite, triangular matrices whose entries are the $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ values for recurrences of type (1) (which he terms *Galton arrays*). These matrices allow one to represent solutions to (1) in terms of simpler recurrences of the same type. Regev and Roichman [13] also obtain (2) with the additional assumption $\beta' = 0$, and they relate special cases of their solution to certain statistics on colored permutations. See also the papers by Mijajlović and Marković [10] and by Cakić [2] where solutions to (1) are given in the case $\beta = \gamma' = 1$, $\gamma = \alpha' = \beta' = 0$ (also, see [3]). Using a partial finite difference approach, Spivey [16, Theorem 6] provides another derivation of (2) and also obtains solutions to (1) for the cases (i) $\alpha = -\beta$, (ii) $\beta = \beta' = 0$, and (iii) $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$. In all of these cases, as with Neuwirth's result (2), nothing more than binomial coefficients, the two kinds of Stirling numbers, and generalized factorials are required to express the solutions and thus are the "fundamental solutions" requested by Graham, Knuth, and Patashnik in these cases.

In this paper, we will consider some combinatorial aspects of recurrence (1). First, we will provide a combinatorial solution to (1) when $\alpha' = 0$ by describing a structure whose cardinality satisfies (1) in this case and then showing that the cardinality is also given by (2). We then use this structure to provide combinatorial proofs of some previous identities satisfied by $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ when $\alpha' = 0$ as well as establish some new ones. Modifying slightly our proof of (2) yields a solution to (1) when $\alpha = -\beta$. We also supply combinatorial proofs of two formulas relating binomial coefficients and the two kinds of Stirling numbers which were previously obtained in [16] by equating three different expressions for the solution of (1) in the case when $\alpha' = \beta' = 0$ and all other weights are unity.

Next, we provide a combinatorial solution to (1) when $\beta = \beta' = 0$. Our expression for the solution in this case differs from the one obtained in [16] using algebraic methods, and in fact it can be shown, bijectively, that the two expressions are equivalent. We also provide in this case a combinatorial proof for an explicit formula of the row sum $\sum_{k=0}^{n} \left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ and extend it to the case when $\beta + \beta' = 0$, which was obtained in [16, Corollary 19] and [12]. Finally, our proof of (2) above may be extended further to ascertain an explicit formula for the solution of a q-version of recurrence (1) when $\alpha' = 0$ and to deduce some identities satisfied by it.

We will use the following notational conventions. Empty sums assume the value 0 and empty products the value 1, with $0^0 = 1$. If m and n are positive integers, then $[m, n] = \{m, m+1, \dots, n\}$ if $m \leq n$, with $[m, n] = \varnothing$ if $m > n$. We will denote the special case $[1, n]$ by $[n]$ if $n \geq 1$, with $[0] = \varnothing$. Let $\mathcal{S}_{n,m}$ be the set of permutations of $[n]$ having m cycles and $\mathcal{S}_n$ be the set of all permutations of $[n]$. Recall that the cardinality of $\mathcal{S}_{n,m}$ is the (signless) Stirling number of the first kind $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$; see, e.g., [18, p. 18]. If one expresses $\sigma \in \mathcal{S}_{n,m}$ as $\sigma = (C_1)(C_2)\cdots(C_m)$, where the smallest letter is first within each cycle C_i and where $\min(C_1) < \min(C_2) < \cdots < \min(C_m)$, then σ is said to be in *standard cycle form*. For example, $\sigma = (134)(257)(69)(8) \in \mathcal{S}_{9,4}$ is in standard cycle form. In what follows, we will compare cycles of some permutation by comparing the sizes of the smallest elements contained within; that is, if (C) and (D) are distinct cycles, then we will say that (C) is smaller than (D) if and only if $\min(C) < \min(D)$.

A *partition* of a finite set is a collection of non-empty, pairwise disjoint subsets, called *blocks*, whose union is the set. The set of all partitions of $[n]$ having exactly m blocks will be denoted by $\mathcal{P}_{n,m}$ whose cardinality is given by the Stirling number of the second kind $\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}$; see, e.g., [18, p. 33]. If one expresses $\pi \in \mathcal{P}_{n,m}$ as $\pi = B_1/B_2/\cdots/B_m$, where $\min(B_1) < \min(B_2) < \cdots < \min(B_m)$, then π is said to be in *standard form*. For example, $\pi = 1, 5, 7/2, 3/4, 6, 8 \in \mathcal{P}_{8,3}$ is in standard form. An *ordered partition* is one in which the blocks themselves are arranged in some order. Note that there are $m!\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}$ ordered partitions of $[n]$ having exactly m blocks, which are synonymous with the surjective functions from $[n]$ to $[m]$.

2. A combinatorial approach

In this section, we provide combinatorial solutions of recurrence (1) in the cases when $\alpha' = 0$ and when $\beta = \beta' = 0$. In the former case, we also deduce some further identities satisfied by $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$.

2.1. A general two-term recurrence

We first provide a combinatorial proof of the following result, which occurs in [12,16].

Theorem 2.1. *Let* $\left|\begin{smallmatrix} n \\ k \end{smallmatrix}\right|$ *denote the array of numbers defined by the recurrence*

$$\left|\begin{matrix} n \\ k \end{matrix}\right| = (\alpha(n-1) + \beta k + \gamma)\left|\begin{matrix} n-1 \\ k \end{matrix}\right| + (\beta' k + \gamma')\left|\begin{matrix} n-1 \\ k-1 \end{matrix}\right| + [n = k = 0], \quad n, k \geq 0. \tag{3}$$