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Complementary cycles in almost regular multipartite tournaments, where one cycle has length four

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ABSTRACT

Let D be a digraph with vertex set $V(D)$ and independence number $\alpha(D)$. If $x \in V(D)$, then the numbers $d^+(x)$ and $d^-(x)$ are the outdegree and indegree of x, respectively. The global irregularity of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) \mid x, y \in V(D)\}.$$

If $i_g(D) = 0$, then D is regular, and if $i_g(D) \le 1$, then D is almost regular. A c-partite tournament is an orientation of a complete c-partite graph.

In 1999, Yeo conjectured that each regular c-partite tournament D with $c \ge 4$ and $|V(D)| \ge 8$ contains a pair of vertex-disjoint directed cycles of lengths 4 and $|V(D)| - 4$. In 2004, Volkmann confirmed this conjecture for $c \ge 5$ and $c = 4$ and $\alpha(D) \ge 4$. As a supplement to this result, we prove in this paper the following theorem.

Let D be an almost regular c-partite tournament with $|V(D)| \ge 8$ such that all partite sets have the same cardinality r. If $c \ge 5$ or $c = 4$ and $r \ge 6$, then D contains a pair of vertex-disjoint directed cycles of lengths 4 and $|V(D)| - 4$.

1. Terminology

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. By a *cycle* or *path* we mean a directed cycle or directed path.

In this paper, all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X.

If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the property that there is no arc from Y to X. By $d^+(X, Y)$ we define the number of arcs going from X to Y.

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x, respectively. The *minimum outdegree* and the *minimum indegree* of D are denoted by $\delta^+(D)$ and $\delta^-(D)$, and the *maximum outdegree* and the *maximum indegree* of D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively. A digraph D is *p-inregular* when $\delta^-(D) = \Delta^-(D) = p$.

The *global irregularity* of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) \mid x, y \in V(D)\},$$

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and the *local irregularity* by $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D. If $i_g(D) = 0$, then D is *regular*, and if $i_g(D) \leq 1$, then D is *almost regular*.

A cycle of length m is an *m-cycle*. A cycle in a digraph D is *Hamiltonian* if it contains all the vertices of D. A digraph is *Hamiltonian* if it contains a Hamiltonian cycle. A set $X \subseteq V(D)$ of vertices is *independent* if the induced subdigraph $D[X]$ has no arcs. The *independence number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D.

A digraph D is *strongly connected* or *strong* if for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k + 1$ vertices is *k-connected* if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ obtained from D by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. A cycle-factor with the minimum number of cycles is called a *minimal cycle-factor*. If x is a vertex of a cycle C, then the *predecessor* and the *successor* of x on C are denoted by x^- and x^+, respectively.

2. Introduction and preliminary results

A digraph D is called *cycle complementary* if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. The problem of complementary cycles in tournaments was almost completely solved by Reid [6] in 1985 and Z. Song [7] in 1993. These authors proved that every 2-connected tournament T on at least 8 vertices has complementary cycles of length t and $|V(T)| - t$ for all $t \in \{3, 4, \ldots, |V(T)| - 3\}$. For c-partite tournaments with $c \geq 4$, there exists the following conjecture.

Conjecture 2.1 (*Yeo [16]*). *A regular c-partite tournament D with $c \geq 4$ and $|V(D)| \geq 8$ has a pair of vertex-disjoint cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \ldots, |V(D)| - 3\}$.*

In 2004 and 2005, Volkmann [9,10] confirmed this conjecture for $t = 4$ and $t = 3$, unless D is a regular 4-partite tournament with two vertices in each partite set. In 2009, He, Korneffel, Meierling, Volkmann and Winzen [4] showed that Conjecture 2.1 is valid for $t = 5$ and $|V(D)| \geq 10$. For more information on complementary cycles in multipartite tournaments we refer the reader to the survey articles [8,11] by Volkmann.

As a supplement to the result in [9], we will prove in this paper the following theorem. Let D be an almost regular c-partite tournament with $|V(D)| \geq 8$ such that all partite sets have the same cardinality r. If $c \geq 5$ or $c = 4$ and $r \geq 6$, then D contains a pair of vertex-disjoint cycles of lengths 4 and $|V(D)| - 4$.

The following results play an important role in our investigations.

Theorem 2.2 (*Moon [5]*). *Let T be a strongly connected tournament. Then every vertex of T is contained in an m-cycle for each $m \in \{3, 4, \ldots, |V(T)|\}$.*

Theorem 2.3 (*Bondy [1]*). *Each strong c-partite tournament with $c \geq 3$ contains an m-cycle for each $m \in \{3, 4, \ldots, c\}$.*

Theorem 2.4 (*Reid [6], Song [7]*). *If T is a 2-connected tournament with $|V(T)| \geq 8$, then T contains two complementary cycles of length t and $|V(T)| - t$ for every $3 \leq t \leq |V(T)|/2$.*

Theorem 2.5 (*Yeo [14]*). *Let D be a $(\lfloor q/2 \rfloor + 1)$-connected multipartite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.*

Theorem 2.6 (*Yeo [14]*). *Let D be a multipartite tournament having a cycle-factor but no Hamiltonian cycle. Then there exists a partite set V^* of D and an indexing $C_1, C_2, \ldots, C_t$ of the cycles of some minimal cycle-factor of D such that for all arcs yx from C_j to C_1 for $2 \leq j \leq t$, it holds that $\{y^+, x^-\} \subseteq V^*$.*

Theorem 2.7 (*Yeo [15]*). *If D is multipartite tournament, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} \right\rceil.$$

Theorem 2.8 (*Yeo [17]*). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. If*

$$i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

Lemma 2.9 (*Yeo [17], Gutin, Yeo [3]*). *A digraph D has no cycle-factor if and only if its vertex set V(D) can be partitioned into four subsets Y, Z, R_1, and R_2 such that*

$$R_1 \rightsquigarrow Y, \qquad (R_1 \cup Y) \rightsquigarrow R_2, \quad \text{and} \quad |Y| > |Z|, \tag{1}$$

where Y is an independent set.