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Note

On spanning cycles, paths and trees

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ABSTRACT

Let G be a simple connected graph with minimum degree δ. Then G is Hamiltonian if it contains a spanning cycle and traceable if it contains a spanning path. The leaf number $L(G)$ of G is defined as the maximum number of end vertices contained in a spanning tree of G. We prove a sufficient condition, depending on $L(G)$ and δ, for G to be Hamiltonian or traceable. Our results, apart from providing a new sufficient condition for Hamiltonicity, settle completely a conjecture of the computer program, Graffiti.pc, instructed by DeLaViña.

1. Introduction

Let $G = (V, E)$ be a connected simple graph. The graph G is *Hamiltonian* if it contains a spanning cycle and *traceable* if it has a spanning path. Results on sufficient conditions for a graph to be Hamiltonian were first presented by Dirac [4] and subsequently reported by several authors (see, for example [7], for an excellent survey). The *leaf number* $L(G)$ of G, whose practical applications in network design are legion, is defined as the maximum number of end vertices contained in a spanning tree of G.

Spanning tree optimization problems naturally appear in many applications, such as in centralized terminal network design and connection routing [5,11]. Usually, the cost of the devices (i.e., software and hardware) associated with each terminal depends on the functionality of the terminal, namely, ending, forwarding or routing a connection. Often since ending terminals do not require message forwarding equipment, devices associated with these terminals, represented by leaf vertices, are cheaper. Thus, if G represents the centralized terminal network, which usually involves other certain constraints such as degree constraints which are either related with the performance of the network or with the availability of some classes of devices, we then ask for a spanning tree of G containing as many leaf vertices as possible.

On the other hand, the existence of a Hamiltonian cycle or path in such networks suffice to solve data communication problems [12]. In the data communication problem, one terminal, the source, has to send data, one datum at a time, to every other terminal in the network, and the data to be sent is different for each destination. Hence if the network G has a Hamiltonian path or cycle, then the source sends, one by one, data along the cycle (path). At each step, each terminal receives a new datum, checks whether it is the destination for the datum, receives the message if so and forwards it to the next terminal in the cycle (path). Clearly, there is no congestion in the data movement, and each terminal receives its message in $O(n)$ time, where n is the number of terminals in G, which is optimal.

It is therefore a natural question to ask for sufficient conditions guaranteeing Hamiltonicity and traceability in a graph of a given leaf number. DeLaViña's computer program, Graffiti.pc (see, for example [2]), which sorts through various graphs and looks for simple relations among parameters, posed the following conjecture and wrote it on the wall [1]:

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Conjecture 1.1 (*Graffiti.pc 190a*)**.** *If G is a simple connected graph with more than one vertex such that $\delta \geq L(G) - 1$, then G is traceable.*

On the Wall II [1], they remarked, "since we thought about this conjecture, it is included on this list". For graphs of minimum degree at least 5, a rather stronger result, which provides a partial solution to Conjecture 1.1, was announced by the author in [9].

Theorem 1.1 (*Mukwembi [9]*)**.** *Let G be a finite connected graph with minimum degree $\delta \geq 5$, and leaf number L. If $\delta \geq L - 1$, then G is Hamiltonian.*

The purpose of this paper is to settle completely Conjecture 1.1. For $\delta \neq 2$, the confirmation of Conjecture 1.1 is a consequence of other known results, among them Theorem 1.1, while the case $\delta = 2$ is more complicated and requires an intricate analysis. The work is organized as follows. In the next section we state, without proof, some known results relevant to our work. In the first part of Section 3 we extend Theorem 1.1 to graphs with minimum δ, $\delta \geq 3$. This is followed by providing examples of graphs with minimum degree δ, $\delta \in \{1, 2\}$, for which Theorem 1.1 does not hold. The last part of Section 3 deals with the more complicated problem, i.e., when $\delta = 2$. We conclude the paper, in Section 4, with a discussion of some new directions for further research.

The notation we use is as follows. Let S be a subset of $V = V(G)$. We denote the subgraph of G induced by S by $G[S]$. The distance, $d_G(u, v)$, between vertices u and v in G is defined as the length of a shortest path joining u and v. A path joining vertices u and v in G is denoted P_{uv}. The neighbourhood, $N_G(u)$, of a vertex u of G is the set $\{x \in V : d_G(x, u) = 1\}$. We denote the degree of a vertex v by $\deg_G(v)$. The second neighbourhood, $N_G^2[u]$, of u is the set $\{x \in V : d_G(x, u) \leq 2\}$. The minimum degree of G, i.e., the smallest value of the degrees of vertices of G, is denoted by $\delta = \delta(G)$. Where there is no danger of confusion, we drop the subscript or argument G. If H is a subgraph of G, we write $H \leq G$.

2. Known results

The determination of the leaf number is known to be NP-hard (see, for example [3]). Of particular interest, in light of for instance centralized terminal networks, is to determine best lower bounds on the leaf number. Lower bounds on the leaf number in terms of other parameters, such as, order, independence number and maximum order of a bipartite graph [2], order and size [3], have been investigated. However, the first result on lower bounds seems to be a statement, without proof, by Storer [12] that every connected cubic graph G with n vertices has $L(G) \geq \frac{n}{4} + 2$. Linial (see, [3]) conjectured, more generally, that every connected graph G with n vertices and minimum degree δ satisfies

$$L(G) \geq \frac{\delta - 2}{\delta + 1} n + c_\delta,$$

where c_δ is a constant depending only on δ. Several authors have researched on this conjecture. Kleitman and West [8] introduced a heavy method, the dead leaves approach, with which they gave a proof of Linial's conjecture for $\delta = 3$ with a best possible $c_\delta = 2$, and hence providing, for the first time, a rigorous proof to Storer's theorem. Precisely, they proved:

Theorem 2.1 (*Kleitman and West [8]*)**.** *If G is a connected simple graph with n vertices and minimum degree at least 3, then $L(G) \geq \frac{1}{4} n + 2$.*

Subsequently, Griggs and Wu [6], using the complicated dead leaves approach, settled Linial's Conjecture for $\delta = 4$ and 5. In this paper, we will make use of one of their theorems.

Theorem 2.2 (*Griggs and Wu [6]*)**.** *If G is a connected simple graph with n vertices and minimum degree at least 4, then $L(G) \geq \frac{2}{5} n + \frac{8}{5}$.*

Ding, Johnson and Seymour [3] introduced a different approach to the study. They do not impose any condition on the individual degrees in the graph; instead, they used order and size to find a lower bound on $L(G)$.

Theorem 2.3 (*Ding, Johnson and Seymour [3]*)**.** *Let G be a connected graph of order n and size m. If $m \geq n + \frac{1}{2} t(t - 1)$ and $n \neq t + 2$, then G has a spanning tree with $>t$ leaves.*

The following simple lemma, which we also use in this paper, was proved in [10].

Lemma 2.1 (*Mukwembi and Munyira [10]*)**.** *Let G be a connected graph and $T' \leq G$ a tree. Then there exists a spanning tree T of G such that $T' \leq T$ and $L(T) \geq L(T')$.*

Finally, we will also find the following folklore result, due to Dirac, handy.

Theorem 2.4 (*Dirac [4]*)**.** *Let G be a connected graph of order n and minimum degree δ. If $\delta \geq \frac{n}{2}$, then G is Hamiltonian.*

3. Results

We begin by presenting very elementary, but handy, observations on Hamiltonicity.