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Bounded edge-connectivity and edge-persistence of Cartesian product of graphs☆

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ABSTRACT

The bounded edge-connectivity $\lambda_k(G)$ of a connected graph G with respect to $k\,(\geq d(G))$ is the minimum number of edges in G whose deletion from G results in a subgraph with diameter larger than k and the edge-persistence $D^+(G)$ is defined as $\lambda_{d(G)}(G)$, where $d(G)$ is the diameter of G. This paper considers the Cartesian product $G_1 \times G_2$, shows $\lambda_{k_1+k_2}(G_1 \times G_2) \geq \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2)$ for $k_1 \geq 2$ and $k_2 \geq 2$, and determines the exact values of $D^+(G)$ for $G = C_n \times P_m$, $C_n \times C_m$, $Q_n \times P_m$ and $Q_n \times C_m$.

1. Introduction

We follow [24] for graph-theoretical terminology and notation not defined here. Throughout this paper, a graph $G = (V, E)$ always means a connected and simple graph (without loops and multiple edges), where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. It is well known that the underlying topology of an interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

Let x and y be two distinct vertices in a graph $G = (V, E)$. The distance $d_G(x, y)$ between x and y is the number of edges in the shortest xy-path, and the diameter of G is $d(G) = \max\{d_G(x, y) : x, y \in V(G)\}$. It is quite natural that, when an interconnection network is modeled by a graph G, the diameter $d(G)$ directly depicts transmission delay of the network if the store-forward time of messages is the same at every vertex. Thus, the diameter is often taken as a measure of efficiency, which is an important parameter to measure the performance of an interconnection network. In order to improve or increase the efficiency of message transmission we need to minimize the diameter of the graph. This is the reason why this concept has received considerable attention in the literature. Many famous graph-theoreticians were interested in this topics, such as Erdős, Rényi, and Sós in [8–10], Alon, Gyárfás, and Ruszinkó [1], Harary [12], Chung [6,7], and so on. The interested reader is referred to the survey paper [3] for early results.

Since some link faults may happen when a network is put into use, it is practically meaningful and important to consider faulty networks. In other words, the removal of some edges in a graph may result in increasing of diameter of the remaining graph, which motivated Chung and Garey [7] to propose the following concept. The edge-fault-tolerant diameter $D_t(G)$ of a t-edge-connected graph G is defined as

$$D_t(G) = \max\{d(G - F) : F \subset E(G), |F| < t\}.$$

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On the other hand, in a real-time system, the message delay must be limited within a given period since any message obtained beyond the bound may be worthless. A natural question is how many faulty links at most can synchronously happen in the network to ensure message delay within the effective bounds. In the language of graph theory, this problem can be stated as follows. At most how many edges can be removed from a graph to ensure no increase of diameter of the remaining graph. In the literature, this question is called the "edge-deletion problem". However, this problem is quite difficult in general, since it has been proved to be NP-complete by Schoone, Bodlaender and van Leeuwen [20].

To investigate further this problem mentioned above, Exoo [11], motivated from Boesch et al. [4], proposed a measure of network vulnerability, called the edge-persistence. The *edge-persistence* $D^+(G)$ of a graph G is the minimum number of edges whose deletion from G increases the diameter of G. For example, $D^+(P_m) = D^+(C_n) = 1$, where P_m is a path of order m and C_n is a cycle of length n. Motivated by Lovász, Neumann-Lara and Plummer [14], Xu [23] generalized this concept to more general case, called the bounded edge-connectivity.

For any positive integer k and $x, y \in V(G)$, the *xy-bounded edge-connectivity* $\lambda_k(G; x, y)$ with respect to k is the minimum number of edges in G whose deletion destroys all xy-paths of length at most k. The *bounded edge-connectivity* of G with respect to k is defined as

$$\lambda_k(G) = \min\{\lambda_k(G; x, y) : x, y \in V(G)\}.$$

Clearly, $\lambda_k(G) \le \delta(G)$, where $\delta(G)$ is the minimum degree of G. If $k \le d(G) - 1$, then $\lambda_k(G) = 0$. Thus, we assume that $k \ge d(G)$ in this paper. Specially, $\lambda_1(G) = 1$ if and only if $G = K_m$ is a complete graph of order $m \ge 2$. It is also clear that $\lambda_k(G) = D^+(G)$ if $k = d(G)$, and $\lambda_{n-1}(G) = \lambda(G)$, the classical edge-connectivity of G, if $n = |V(G)|$. Thus, the bounded edge-connectivity is a generalization of both the edge-persistence and the classical edge-connectivity.

In [23], Xu established the relationships between $\lambda_k(G)$ and $D_t(G)$ as follows. For any connected graph G,

(a) $\lambda_k(G) = t \Leftrightarrow D_t(G) \le k < D_{t+1}(G)$ if G is $(t+1)$-edge-connected, or

(b) $D_t(G) = k \Leftrightarrow \lambda_{k-1}(G) < t \le \lambda_k(G)$ if G is t-edge-connected.

The three parameters $\lambda_k(G)$, $D_t(G)$ and $D^+(G)$ can be viewed as important measures of the vulnerability of communication networks modeled as graphs and, thus, have received much research attention in the past years, see, for example, [4,5,7,11, 12,15–18,20–23,25].

We consider the Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2. For graphs G_1 and G_2, the Cartesian product $G_1 \times G_2$ is the graph with vertex-set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge-set $E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2) | x_1 = y_1$ and $x_2y_2 \in E(G_2)$ or $x_2 = y_2$ and $x_1y_1 \in E(G_1)\}$.

It is well known that the Cartesian product is an important research topic in graph theory (see, e.g., [13]). It is also well known that, for designing large-scale interconnection networks, the Cartesian product is an important method to obtain large graphs from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. The Cartesian product preserves many nice properties such as regularity, existence of Hamilton cycles and Euler circuits, and transitivity of the initial graphs (see, e.g., [23]). In fact, many well-known networks can be constructed by the Cartesian products of some simple graphs. For example, the n-dimensional hypercube Q_n is the Cartesian product of n complete graphs of order 2, a torus is the Cartesian product of two cycles, and a mesh is the Cartesian product of two paths.

What we are interested in is the bounded edge-connectivity and edge-persistence of the Cartesian product of graphs. Graham and Harary [12] showed $D^+(Q_n) = n - 1$; Sung and Wang [21] investigated $D^+(C_m \times C_n)$, etc., and conjectured $D^+(G_1 \times G_2) \ge \max\{D^+(G_1), D^+(G_2)\} + 1$.

In this paper, we first establish a lower bound of λ_k for the Cartesian product $G_1 \times G_2$, that is, $\lambda_{k_1+k_2}(G_1 \times G_2) \ge \lambda_{k_1}(G_1) + \lambda_{k_2}(G_2)$ for $k_i \ge 2$, $i = 1, 2$. As an immediate consequence, we obtain $D^+(G_1 \times G_2) \ge D^+(G_1) + D^+(G_2)$ if $d(G_i) \ge 2$ for $i = 1, 2$. This lower bound is tight, and gives an affirmative answer to the above-mentioned conjecture of Sung and Wang if the diameters of both G_1 and G_2 are at least two. Then we determine $D^+(C_n \times P_m) = 1$ for $n = 3$ and 2 for $n \ge 4$; $D^+(C_n \times C_m) = 2$ if $n = 3$ or $m = 3$ or both n and m are odd, 3 otherwise. Lastly, we determine $D^+(Q_n \times P_m) = n$ for $n \ge 2$ and $m \ge 2$; $D^+(Q_n \times C_m) = n$ for $m = 3$, $n + 1$ for $m \ge 4$. These results correct some inaccurate results on $D^+(C_n \times C_m)$ in [21].

The rest of the paper is organized as follows. In Section 2 we establish the lower bound of $\lambda_{k_1+k_2}(G_1 \times G_2)$. The results on $D^+(G_1 \times G_2)$ for some P_n, C_n and Q_n are presented in Section 3. The conclusions and remarks are in Section 4.

2. Bounded edge-connectivity

For a vertex $x \in V(G_1)$ and a subgraph $H \subseteq G_2$, we use xH to denote the subgraph of $G_1 \times G_2$ induced by $\{x\} \times V(H)$. Similarly, for a vertex $y \in V(G_2)$, a subgraph $H \subseteq G_1$, Hy denotes the subgraph of $G_1 \times G_2$ induced by $V(H) \times \{y\}$. For a path $P = x_1 \cdots x_i \cdots x_j \cdots x_n$ in G, $P(x_i, x_j)$ denotes the section $x_i \cdots x_j$ of P. For the sake of convenience, we will denoted P as

$$P = x_1 \xrightarrow{P(x_1,x_i)} x_i \xrightarrow{P(x_i,x_j)} x_j \xrightarrow{P(x_j,x_n)} x_n.$$

The symbol $\varepsilon(P)$ denotes the length of P, which is the number of edges in P.

Now, we state our main result in this paper.