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Antibandwidth and cyclic antibandwidth of Hamming graphs

Stefan Dobrev [a], Rastislav Královič [b], Dana Pardubská [b], L'ubomír Török [c,*], Imrich Vrt'o [a]

[a] Institute of Mathematics, Slovak Academy of Sciences, Dúbravská 9, 841 04 Bratislava, Slovakia
[b] Department of Computer Science, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia
[c] Institute of Mathematics and Computer Science, Slovak Academy of Sciences and Matej Bel University, Ďumbierska 1, 974 11, Banská Bystrica, Slovakia

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ABSTRACT

The antibandwidth problem is to label vertices of a graph $G(V, E)$ bijectively by integers $0, 1, \ldots, |V| - 1$ in such a way that the minimal difference of labels of adjacent vertices is maximized. In this paper we study the antibandwidth of Hamming graphs. We provide labeling algorithms and tight upper bounds for general Hamming graphs $\Pi_{k=1}^{d} K_{n_k}$. We have exact values for special choices of n_i's and equality between antibandwidth and cyclic antibandwidth values. Moreover, in the case where the two largest sizes of n_i's are different we show that the Hamming graph is multiplicative in the sense of [9]. As a consequence, we obtain exact values for the antibandwidth of p isolated copies of this type of Hamming graphs.

1. Introduction

The antibandwidth problem is to label vertices of a graph $G(V, E)$ bijectively by integers $0, 1, \ldots, |V| - 1$ in such a way that the minimal difference of labels of adjacent vertices is maximized. The maxmin difference is called the antibandwidth of G.

This problem was originally introduced in [15] in connection with multiprocessors scheduling problems. Another motivation comes from the area of frequency assignment problem [10] and obnoxious facility location problems [5]. This problem is a dual one to the well-known bandwidth minimization problem [6] and also belongs to the large family of graph labeling problems [8]. The antibandwidth problem is *NP*-complete for general graphs. The question "Is $\text{ab}(G) \geq 2$?" is equivalent to deciding whether the complement of G contains a Hamiltonian path. So far there exist polynomial algorithms for 3 classes of graphs: the complements of interval, arborescent comparability and threshold graphs [7,14]. Recently, new heuristic methods have appeared in literature [2,3]. Known results on antibandwidth include exact values and tight bounds for paths, cycles, special trees [4,18,22], meshes [20,19], tori and hypercubes [17,21]. In the area of graph drawings a problem called the "maximum differential graph coloring problem" recently appeared. This problem is basically the same as the antibandwidth problem [13].

The cyclic antibandwidth is a natural and typical extension of the original problem when the differences are computed as distances "around cycle". The value of the cyclic antibandwidth is determined for meshes, toroidal meshes and hypercubes in [17].

In this paper we provide antibandwidth and cyclic antibandwidth values for d-dimensional Hamming graphs. This class of graphs is interesting because of its connection to the area of the error-correcting codes [11] and association schemes. Particularly, we show that if $2 \leq n_1 \leq n_2 \leq \cdots \leq n_{d-1} < n_d$, then

$$\text{ab}(\Pi_{k=1}^{d} K_{n_k}) = n_1 n_2 \cdots n_{d-1}.$$

* Corresponding author. Fax: +421 48 415 10 12.
E-mail addresses: Stefan.Dobrev@savba.sk (S. Dobrev), kralovic@dcs.fmph.uniba.sk (R. Královič), pardubsk@dcs.fmph.uniba.sk (D. Pardubská), torok@savbb.sk (L. Török), vrto@savba.sk (I. Vrt'o).

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We prove equality between the antibandwidth and the cyclic antibandwidth for most of the Hamming graphs. Moreover, in the case where the two largest sizes of n_i's are different we show that the Hamming graph is multiplicative in the sense of [9]. As a consequence we obtain exact values for the antibandwidth of p isolated copies of this type of Hamming graphs. Finally, let us mention that the dual problem—bandwidth of the Hamming graph—was studied in [1,12].

2. Preliminaries

For a nonempty graph $G = (V, E)$, let f be a bijective labeling

$$f : V \to \{0, 1, 2, 3, \dots, |V| - 1\}.$$

Define the antibandwidth of G according to f by

$$\mathrm{ab}(G, f) = \min_{uv\in E} |f(u) - f(v)|.$$

The antibandwidth of G is defined by

$$\mathrm{ab}(G) = \max_{f} \mathrm{ab}(G, f),$$

where the maximum is taken over all bijective labelings f. Define the cyclic antibandwidth of a connected graph G according to f by

$$\mathrm{cab}(G, f) = \min_{uv\in E}\{|f(u) - f(v)|, |V| - |f(u) - f(v)|\}.$$

The cyclic antibandwidth of G is defined by

$$\mathrm{cab}(G) = \max_{f} \mathrm{cab}(G, f),$$

with maximum taken over all bijective labelings f.

The d-dimensional Hamming graph $\Pi_{k=1}^{d} K_{n_k}$ is defined by the Cartesian product of d complete graphs K_{n_k}, for $k = 1, 2, \dots, d$. The vertices of $\Pi_{k=1}^{d} K_{n_k}$ are d-tuples $(i_1, i_2, \dots, i_d)$, where $i_k \in \{0, 1, 2, \dots, n_k - 1\}$. Two vertices $(i_1, i_2, \dots, i_d)$ and $(j_1, j_2, \dots, j_d)$ are adjacent iff the two d-tuples differ in precisely one coordinate. In case $n_k = n$, for all k, we denote the graph as K_n^d. Define the value of N_k as follows. Set $N_0 = 1$ and for $k = 1, 2, \dots, d$, denote $N_k = n_1 n_2 \cdots n_k$.

3. Antibandwidth of Hamming graphs

In this section we prove our main result: exact and tight bounds for the antibandwidth of Hamming graphs.

Theorem 3.1. *For $d \geq 2$ and $2 \leq n_1 \leq n_2 \leq \cdots \leq n_d$,*

$$\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) = n_1 n_2 \cdots n_{d-1}, \quad \text{if } n_{d-1} \neq n_d,$$
$$\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) = n_1 n_2 \cdots n_{d-1} - 1, \quad \text{if } n_{d-1} = n_d \text{ and } n_{d-2} \neq n_{d-1},$$

and

$$n_1 n_2 \cdots n_{d-1} - \min\{n_1 n_2 \cdots n_{d-2}, n_{q+1} \cdots n_{d-1}\} \leq \mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) \leq n_1 n_2 \cdots n_{d-1} - 1,$$

where $n_{d-2} = n_{d-1} = n_d$, $d \geq 3$ and q is the minimal index such that $q \leq d - 2$ and $n_q = n_d$.

Proof. *Upper bound*. Let $\alpha(G)$ denote the size of the largest independent set of a graph G. From [16] we have $\mathrm{ab}(G) \leq \alpha(G)$. We show that $\alpha(\Pi_{k=1}^{d} K_{n_k}) \leq N_{d-1}$, which will prove a general upper bound. Partition the vertices of $\Pi_{k=1}^{d} K_{n_k}$ into N_{d-1} sets. For fixed a_i, $0 \leq a_i \leq n_i - 1$, $0 \leq i \leq d - 1$, let the corresponding set be $\{(a_1, a_2, \dots, a_{d-1}, x_d) | 0 \leq x_d \leq n_d - 1\}$. Given any independent set I of $\Pi_{k=1}^{d} K_{n_k}$, realize that every partition set contains at most one vertex from I, which proves the claim.

In case $n_{d-1} = n_d$ we can slightly improve the general upper bound. Consider any labeling function f. We may imagine that vertices of $\Pi_{k=1}^{d} K_{n_k}$ are placed on a real line into integer points $0, 1, 2, \dots, N_d - 1$, such that a vertex v, labeled by $f(v)$, is placed at the position $f(v)$. Every vertex of $\Pi_{k=1}^{d} K_{n_k}$ belongs to two cliques: K_{n_d} and $K_{n_{d-1}}$, whose intersection is precisely that vertex. Consider the vertex placed at the position $\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) - 1$ and the corresponding cliques K_{n_d} and $K_{n_{d-1}}$. Clearly, all vertices of these cliques must lie in the interval $[\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) - 1, N_d - 1]$. Because $n_{d-1} = n_d$, one of the cliques, say K_{n_d}, must lie in a shorter interval, otherwise the two cliques would have two vertices in common. Hence

$$\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) \leq \frac{N_d - 2 - (\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) - 1)}{n_d - 1} \leq N_{d-1} - \frac{1}{n_d - 1},$$

which implies $\mathrm{ab}(\Pi_{k=1}^{d} K_{n_k}) \leq N_{d-1} - 1$.