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On generalized zero divisor graph of a poset

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ABSTRACT

In this paper, we introduce the generalized ideal based zero divisor graph of a poset P, denoted by $\widehat{G_I(P)}$. A representation theorem is obtained for generalized zero divisor graphs. It is proved that a graph is complete r-partite with $r \geqslant 2$ if and only if it is a generalized zero divisor graph of a poset. As a consequence of this result, we prove a form of a Beck's Conjecture for generalized zero divisor graphs of a poset. Further, it is proved that a generalized zero divisor graph $\widehat{G_{\{0\}}(P)}$ of a section semi-complemented poset P with respect to the ideal (0] is a complete graph.

1. Introduction

The idea of a zero divisor graph was introduced by Beck in [2] to investigate the interplay between ring theoretic properties and graph theoretic properties. The concept of zero divisor graph is also well studied in other algebraic structures such as semigroups, semilattices see Nimbhorkar et al. [12]. In [4], Halaš and Jukl (see also Halaš and Länger [5]) introduced the zero divisor graph of a poset (qoset). Recently, the zero divisor graph of a poset with respect to an ideal has been studied in Joshi [7]. For more details on zero divisor graphs of lattices and posets, see [8–10].

In this paper, we introduce the generalized ideal based zero divisor graph of a poset P, denoted by $\widehat{G_I(P)}$. A representation theorem is obtained for generalized zero divisor graphs. It is proved that a graph is complete r-partite with $r \geqslant 2$ if and only if it is a generalized zero divisor graph of a poset. As a consequence of this result, we prove a form of a Beck's Conjecture for generalized zero divisor graphs of a poset. Further, it is proved that a generalized zero divisor graph $\widehat{G_{\{0\}}(P)}$ of a section semi-complemented poset P with respect to the ideal (0] is a complete graph.

2. Generalized zero divisor graph of a poset

Let P be a non-empty set. A *partial order* on P is a binary relation $\leqslant$ on P such that, it is *reflexive, antisymmetric* and *transitive*. A non-empty set P equipped with a partial order relation "$\leqslant$" is called a *partially ordered set*, in short a *poset*, and denoted by $(P; \leqslant)$, if it is necessary to specify the order relation, otherwise it will be denoted by P. The dual relation of $\leqslant$ is denoted by $\geqslant$.

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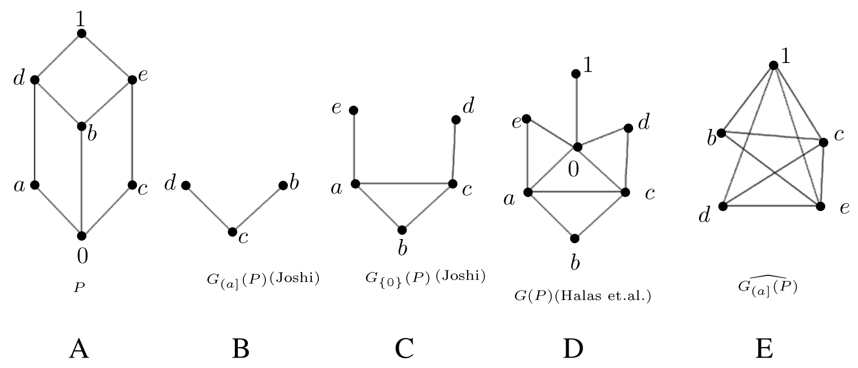

Fig. 1. Poset P and its different zero divisor graphs.

We begin with necessary concepts and terminology in a poset P. Let $A \subseteq P$. The set $A^u = \{x \in P \mid x \geqslant a$, for every $a \in A\}$ is called the *upper cone* of A. Dually, we have the concept of the *lower cone* A^ℓ of A. The upper cone $\{a\}^u$ is simply denoted by a^u and $\{a, b\}^u$ is denoted by $(a, b)^u$. Similar notations are used for lower cones. Further, for $A, B \subseteq P$, $\{A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$, the set $\{A \cup \{x\}\}^u$ is denoted by $\{A, x\}^u$. Similar notations are used for lower cones. By $A^{u\ell}$, we mean $\{A^u\}^\ell$. We note that, for $A, B \subseteq P$, $A \subseteq A^{u\ell}$ and $A \subseteq A^{\ell u}$, further if $A \subseteq B$, then $B^\ell \subseteq A^\ell$ and $B^u \subseteq A^u$. Moreover, $A^{\ell u \ell} = A^\ell$, $A^{u\ell u} = A^u$ and $\{a^u\}^\ell = \{a\}^\ell = a^\ell$.

A non-empty subset I of a poset P is called a *semi-ideal*, if for $x \in I$, $y \in P$, $y \leqslant x$ implies $y \in I$. A proper semi-ideal I of a poset P is called a *prime semi-ideal*, if $(a, b)^\ell \subseteq I$ implies $a \in I$ or $b \in I$; Venkatanarsimhan [16].

A non-empty subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a, b)^{u\ell} \subseteq I$. An ideal $I \neq P$ is called *prime* if $(a, b)^\ell \subseteq I$ implies either $a \in I$ or $b \in I$. Dually, we have the concepts of filters and prime filters. Ideals in posets have been treated also in Halaš [3]. For each $a \in L$, the set $(a] = \{x \in P \mid x \leqslant a\}$ is an ideal of L; it is known as the *principal ideal* generated by a and dually, $[a)$ is a *principal filter*.

Definition 2.1 (*Joshi [7]*). Let I be an ideal of a poset P. We associate with I an undirected graph $G_I(P)$ called *the zero divisor graph of P with respect to the ideal I* as follows: The set of vertices of $G_I(P)$ is $V(G_I(P)) = \{x \in P \setminus I \mid (x, y)^\ell \subseteq I$, for some $y \in P \setminus I\}$ and two distinct vertices x, y are adjacent if and only if $(x, y)^\ell \subseteq I$. When $I = \{0\}$ then the zero divisor graph is denoted by $G_{\{0\}}(P)$.

Halaš and Jukl [4] also studied the concept of a zero divisor graph of a poset P with 0 denoted by $G(P)$ in which the vertex set of the graph is the set of elements of P and two elements $x, y \in P$ are adjacent if and only if $(x, y)^\ell = \{0\}$.

Definition 2.2. Let I be an ideal(semi-ideal) of a poset P. The set $Z_I(P)$ is called *the set of generalized zero divisors with respect to I* of P and is given by $Z_I(P) = \{x \in P \setminus I \mid (x_1, y_1)^\ell \subseteq I$, for some $x_1 \in (x] \setminus I$ and for some $y_1 \notin I\}$.

Remark 2.3. It is easy to observe that if I is a prime semi-ideal then $Z_I(P) = \emptyset$. As such, henceforth, we now consider I as a non-prime ideal(semi-ideal) of a poset P. Then it is easy to observe that $Z_I(P) = P \setminus I$. For this, let $a \in P \setminus I$. Since I is non-prime, there exist $b, c \notin I$ such that $(b, c)^\ell \subseteq I$. If either $(a, b)^\ell \subseteq I$ or $(a, c)^\ell \subseteq I$, then $a \in Z_I(P)$ and we are through. Suppose on the contrary that $(a, b)^\ell \nsubseteq I$ and $(a, c)^\ell \nsubseteq I$. Hence we have $x \in (a, b)^\ell$, $y \in (a, c)^\ell$ such that $x, y \notin I$. But then $(x, y)^\ell \subseteq I$, as $x \leqslant b$, $y \leqslant c$ and $(b, c)^\ell \subseteq I$. In this case also $a \in Z_I(P)$, as $x \in (a] \setminus I$, $y \notin I$ and $(x, y)^\ell \subseteq I$. Thus $Z_I(P) = P \setminus I$.

P. Dheena and B. Elavarasan [13] defined the generalized zero divisor graph of near rings. Analogously, we introduce the concept of a generalized zero divisor graph of a poset with respect to an ideal(semi-ideal) as follows:

Definition 2.4. Let I be a non-prime ideal(semi-ideal) of a poset P. We associate an undirected graph called the *generalized zero divisor graph* of P with respect I, denoted by $\widehat{G_I(P)}$, in which the set of vertices $V(\widehat{G_I(P)}) = Z_I(P) = \{x \in P \setminus I \mid (x_1, y_1)^\ell \subseteq I$, for some $x_1 \in (x] \setminus I$ and for some $y_1 \notin I\}$ and two distinct vertices x and y are adjacent if and only if $(x_1, y_1)^\ell \subseteq I$, for some $x_1 \in (x] \setminus I$ and for some $y_1 \in (y] \setminus I$.

Henceforth, by $\widehat{G_I(P)}$, we mean a generalized zero divisor graph of a poset P with respect to a non-prime ideal(semi-ideal) I and hence from Remark 2.3, it is clear that $V(\widehat{G_I(P)}) = Z_I(P) \neq \emptyset$.

Now, we illustrate the concept of zero divisor graph and generalized zero divisor graph with an example. In Fig. 1, the poset P and its zero divisor graphs are depicted.

We recall the following concepts from graph theory, see Harary [6] and D.B. West [17].

Definition 2.5. Let G be a graph. Let x, y be distinct vertices in G. We denote by $d(x, y)$ the length of a shortest path from x to y (if it exists) and put $d(x, y) = \infty$ otherwise we write $d(x, x) = 0$ for $x \in V(G)$. The *diameter* of G is denoted by