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Algorithmic aspects of the k-domination problem in graphs[☆]

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ABSTRACT

For a positive integer k, a *k-dominating set* of a graph G is a subset $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least k vertices in D. The *k-domination problem* is to determine a minimum k-dominating set of G. This paper studies the k-domination problem in graphs from an algorithmic point of view. In particular, we present a linear-time algorithm for the k-domination problem for graphs in which each block is a clique, a cycle or a complete bipartite graph. This class of graphs includes trees, block graphs, cacti and block-cactus graphs. We also establish NP-completeness of the k-domination problem in split graphs.

1. Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless and without multiple edges. Domination is a core NP-complete problem in graph theory and combinatorial optimization. It has many applications in the real world such as location problems, sets of representatives, social network theory, etc.; see [3,12] for more interesting applications. A vertex is said to *dominate* itself and all of its neighbors. A *dominating set* of a graph G is a subset D of $V(G)$ such that every vertex not in D is dominated by at least one vertex in D. The *domination number* $\gamma(G)$ of G is the minimum size of a dominating set of G. The *domination problem* is to find a minimum dominating set of a graph.

It is well-known that given any minimum dominating set D of a graph G, one can always remove two edges from G such that D is no longer a dominating set for G [12, p. 184]. The idea of dominating each vertex multiple times is naturally considered. One of such generalizations is the concept of *k-domination*, introduced by Fink and Jacobson in 1985 [10]. For a positive integer k, a *k-dominating set* of a graph G is a subset $D \subseteq V(G)$ such that every vertex not in D is dominated by at least k vertices in D. The *k-domination number* $\gamma_k(G)$ of G is the minimum size of a k-dominating set of G. The *k-domination problem* is to determine a minimum k-dominating set of a graph. The special case when $k = 1$ is the ordinary domination.

Many of the k-domination results in the literature focused on finding bounds on the number $\gamma_k(G)$. In particular, bounds in terms of order, size, minimum degree, maximum degree, domination number, independence number, k-independence number, and matching number were extensively studied [2,4,6,7,9–11,16]; also see the recent survey paper [5].

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On the complexity side of the k-domination problem, Jacobson and Peters showed that the k-domination problem is NP-complete for general graphs [14] and gave linear-time algorithms to compute the k-domination number of trees and series–parallel graphs [14]. The k-domination problem remains NP-complete in bipartite graphs or chordal graphs [1]. More complexity results for the k-domination problem are desirable.

In this paper, we explore efficient algorithms for the k-domination problem in graphs. In particular, we present a linear-time algorithm for the k-domination problem in graphs in which each block is a clique, a cycle or a complete bipartite graph. This class of graphs include trees, block graphs, cacti and block-cactus graphs. We also show that the k-domination problem remains NP-complete in split graphs, a subclass of chordal graphs.

2. Preliminaries

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex v, the *open neighborhood* is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of a vertex v in G is the number of edges incident to v.

The *subgraph of G induced by* $S \subseteq V$ is the graph $G[S]$ with vertex set S and edge set $\{uv \in E : u, v \in S\}$. In a graph $G = (V, E)$, the *deletion of* $S \subseteq V$ *from* G, denoted by $G - S$, is the graph $G[V \setminus S]$. For a vertex v in G, we write $G - v$ for $G - \{v\}$.

In a graph, a *stable set* (or *independent set*) is a set of pairwise nonadjacent vertices, and a *clique* is a set of pairwise adjacent vertices. A *forest* is a graph without cycles. A *tree* is a connected forest. A *leaf* of a graph is a vertex with degree one. A vertex v is a *cut-vertex* if the number of connected components is increased after removing v. A *block* of a graph is a maximal connected subgraph without any cut-vertex. An *end-block* of a graph is a block containing at most one cut-vertex. A *block graph* is a graph whose blocks are cliques. A *cactus* is a connected graph whose blocks are either an edge or a cycle. A *block-cactus graph* is a graph whose blocks are cliques or cycles.

3. The labeling method for *k*-domination

Labeling techniques are widely used in the literature for solving the domination problem and its variants [3,8,13,15]. For k-domination, we employ the following labeling method which is similar to that in [15]. Given a graph G, a *k-dom assignment* is a mapping L that assigns each vertex v in G a two-tuple label $L(v) = (L_1(v), L_2(v))$, where $L_1(v) \in \{B, R\}$, and $L_2(v)$ is a nonnegative integer. Here a vertex v with $L_1(v) = R$ is called a *required vertex*; a vertex v with $L_1(v) = B$ is called a *bound vertex*. An *L-dominating set* of G is a subset $D \subseteq V(G)$ such that

- if $L_1(v) = R$, then $v \in D$, and
- if $L_1(v) = B$, then either $v \in D$ or $|N(v) \cap D| \geq L_2(v)$.

That is, D contains all required vertices, and for each bound vertex v not in D, v is adjacent to at least $L_2(v)$ vertices in D. The *L-domination number* $\gamma_L(G)$ is the minimum size of an L-dominating set in G, such set is called a γ_L-set of G. Notice that if $L(v) = (B, k)$ for all $v \in V(G)$, then $\gamma_L(G) = \gamma_k(G)$. Thus an algorithm for $\gamma_L(G)$ gives $\gamma_k(G)$.

Lemma 1. *Suppose G is a graph with a k-dom assignment* $L = (L_1, L_2)$. *For a vertex* v *in G, let* $G' = G - v$ *and let* L' *be the restriction of L on* $V(G')$ *with the modification that* $L'_2(u) = \max\{L_2(u) - 1, 0\}$ *for* $u \in N(v)$. *If* $L_1(v) = R$ *or* $L_2(v) > \deg(v)$, *then* $\gamma_L(G) = \gamma_{L'}(G') + 1$.

Proof. Suppose D' is a $\gamma_{L'}$-set of G'. Set $D = D' \cup \{v\}$. Since L' is the restriction of L on $V(G')$ with the modification on $L'_2(u)$ and $L_2(u) \leq L'_2(u) + 1$ for $u \in N(v)$, D is clearly an L-dominating set of G. Thus $\gamma_L(G) \leq |D| = |D'| + 1 = \gamma_{L'}(G') + 1$.

Conversely, suppose D is a γ_L-set of G. By the assumption that $L_1(v) = R$ or $L_2(v) > \deg(v)$, v must be included in D. Set $D' = D \setminus \{v\}$. As L' is the restriction of L on G' with the modification on $L'_2(u)$ for $u \in N(v)$, D' is an L'-dominating set of $V(G')$. Hence $\gamma_{L'}(G') + 1 \leq |D'| + 1 = |D| = \gamma_L(G)$. □

This and the following lemma provide an alternative algorithm for the k-domination problem in trees.

Lemma 2. *Suppose G is a graph with a k-dom assignment* $L = (L_1, L_2)$. *For a leaf* v *of G adjacent to u, let* $G' = G - v$ *and let* L' *be the restriction of L on* $V(G')$ *with the modification described below.*

(1) *If* $L(v) = (B, 1)$, *then* $\gamma_L(G) = \gamma_{L'}(G')$, *where* $L'_1(u) = R$.
(2) *If* $L(v) = (B, 0)$, *then* $\gamma_L(G) = \gamma_{L'}(G')$.

Proof. (1) Suppose D' is a $\gamma_{L'}$-set of G'. Since $L'_1(u) = R$, we have $u \in D'$. Then D' is an L-dominating set of G as $|N(v) \cap D'| \geq 1 = L_2(v)$. Thus $\gamma_L(G) \leq |D'| = \gamma_{L'}(G')$.

Conversely, suppose D is a γ_L-set of G. Since $L_2(v) = 1$, either u or v must be included in D. Then clearly $D' = (D \setminus \{v\}) \cup \{u\}$ is an L'-dominating set of G'. Hence $\gamma_{L'}(G') \leq |D'| \leq |D| = \gamma_L(G)$.

(2) Suppose D' is a $\gamma_{L'}$-set of G'. Since L' is the restriction of L on G' and $|N(v) \cap D'| \geq 0 = L_2(v)$, it is clear that D' is an L-dominating set of G. Thus $\gamma_L(G) \leq |D'| = \gamma_{L'}(G')$.

Conversely, suppose D is a γ_L-set of G. If $v \notin D$, then $D' = D$ is an L'-dominating set of G'. If $v \in D$, then $D' = (D \setminus \{v\}) \cup \{u\}$ is an L'-dominating set of G'. Hence $\gamma_{L'}(G') \leq |D'| \leq |D| = \gamma_L(G)$. □