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Hamiltonian claw-free graphs involving minimum degrees

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ABSTRACT

Favaron and Fraisse proved that any 3-connected claw-free graph H with order n and minimum degree $\delta(H) \geq \frac{n+38}{10}$ is hamiltonian [O. Favaron and P. Fraisse, Hamiltonicity and minimum degree in 3-connected claw-free graphs, J. Combin. Theory B 82 (2001) 297–305]. Lai, Shao and Zhan showed that if H is a 3-connected claw-free graph of order $n \geq 196$, and if $\delta(H) \geq \frac{n+6}{10}$, then H is hamiltonian [H.-J. Lai, Y. Shao and M. Zhan, Hamiltonicity in 3-connected claw-free graphs, J. Combin. Theory B 96 (2006) 493–504]. In this paper, we improve the two results above and prove that if H is a 3-connected claw-free graph of order $n \geq 363$, and if $\delta(H) \geq \frac{n+34}{12}$, then either H is hamiltonian, or the Ryjáček's closure $cl(H)$ of H is the line graph of one of the graphs obtained from the Petersen graph $\mathcal{P}_{10}$ by adding at least one pendant edge at each vertex v_i of $\mathcal{P}_{10}$ or by replacing exactly one vertex v_i of $\mathcal{P}_{10}$ with $\bar{K}_{2,p}$ $(p \geq 2)$ and adding at least one pendant edge at all other nine vertices $v_j \notin V - \{v_i\}$ of $\mathcal{P}_{10}$, and then by subdividing m edges of $\mathcal{P}_{10}$ for $m = 0, 1, 2, \ldots, 15$, where $\bar{K}_{2,p}$ is a connected bipartite graph.

1. Introduction

We only consider loopless finite simple graphs, and use [1] for terminology and notations not defined here. A graph G is *eulerian* if G is connected and every vertex of G is of even degree. A *circuit* C of a graph G is a connected eulerian subgraph of G. A cycle is a connected circuit with all vertices of degree 2. The *minimum degree* and the *edge independence number* of G are denoted by $\delta(G)$ (or δ) and $\alpha'(G)$, respectively. An edge $e = uv$ is called a *pendant edge* if either $d_G(u) = 1$ or $d_G(v) = 1$. A subgraph H of G (denoted by $H \subseteq G$) is *dominating* if $G - V(H)$ is edgeless. For $x \in V(G)$, let $N_H(x) = \{v \in V(H) : vx \in E(G)\}$ and $d_H(x) = |N_H(x)|$. If $S \subseteq V(G)$, $G[S]$ is the subgraph induced in G by S. A vertex $v \in G$ is called a *locally connected vertex* if $G[N_G(v)]$ is connected. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let $N_H(A) = \cup_{v \in A} N_H(v)$, $E_G[A, B] = \{uv \in E(G) : u \in A, v \in B\}$, and $G - A = G[V(G) - A]$. When $A = \{v\}$, we use $G - v$ for $G - \{v\}$. If $H \subseteq G$, then for an edge subset $X \subseteq E(G) - E(H)$, we write $H + X$ for $G[E(H) \cup X]$. For an integer $i \geq 1$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$.

We write $\bar{K}_{|X|,|Y|}$ for a connected bipartite graph with disjoint vertex sets X and Y, and $K_{|X|,|Y|}$ for a complete bipartite graph. If $|X| = 1$ and $|Y| \geq 2$, then $K_{1,|Y|}$ is called a star, and the vertex of X is called the center of the star. If $|X| = 1$ and $|Y| = 3$, then $K_{1,3}$ is called a claw. A graph H is claw-free if it does not contain $K_{1,3}$ as an induced subgraph. A graph H is triangle-free if it does not contain a cycle of length 3 as an induced subgraph.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Obviously, a line graph is claw-free. Let H be the line graph $L(G)$ of a graph G. Then $|V(H)| = |E(G)|$ and $\delta(H) = \min\{d_G(x) + d_G(y) - 2 : xy \in E(G)\}$. If $L(G)$ is k-connected, then G is *essentially k-edge-connected*, which means that the only edge-cut sets of G having less than k edges are the sets of edges incident with some vertex of G.

Let X be a subset of $E(G)$. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If K is a subgraph of G, then we write G/K for $G/E(K)$. If K is a

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connected subgraph of G, and if v_K is the vertex in G/K onto which K is contracted, then K is called the *preimage* of v_K, and is denoted by $PI(v_K)$. A vertex v in a contraction G/K of G is *nontrivial* if $PI(v)$ has at least one edge from $E(G/K) - E(G)$. A vertex v in a contraction G/K of G is *trivial* if $PI(v)$ contains no any edge belonging to $E(G/K) - E(G)$.

Subdividing the edge uv of a graph G means that the edge uv is replaced by a path uxv of length 2, where $x \notin V(G)$ is a new vertex, and is called the subdivision vertex of the edge uv.

Harary and Nash-Williams [4] showed that there is a close relationship on hamiltonian cycles between a graph and its line graph.

Theorem 1.1 (*Harary and Nash-Williams [4]*)**.** *The line graph $L(G)$ of a graph G is hamiltonian if and only if G has a dominating eulerian subgraph.*

Ryjáček [13] defined the *closure* $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H, as long as this is possible. Note that this operation preserves the claw-freeness of the original graph. Ryjáček [13] proved the following result.

Theorem 1.2 (*Ryjáček [13]*)**.** *Let H be a claw-free graph and $cl(H)$ its closure. Then*

(i) *$cl(H)$ is well-defined, and $\kappa(cl(H)) \geq \kappa(H)$,*
(ii) *there is a triangle-free graph G such that $cl(H) = L(G)$,*
(iii) *both graphs H and $cl(H)$ have the same circumference.*

A graph H with $H = cl(H)$ is called closed. Many researchers are interested in studying the Hamiltonicity of claw-free graphs, and many works have been done to give sufficient conditions for a claw-free graph to be hamiltonian in terms of its minimum degrees. These conditions depend on the connectivity $\kappa(H)$. Matthews and Sumner [12] conjectured that every 4-connected claw-free graph H is hamiltonian. If $\kappa(H) = 3$, there are a lot of non-hamiltonian claw-free graphs. For example, M. Li [7] showed that every 3-connected claw-free graph H of order $n \leq 5\delta(H) - 5$ is hamiltonian. G. Li et al. [11] and M. Li [8], respectively, improved the result and obtained that every 3-connected claw-free graph H of order $n \leq 6\delta(H) - 9$ is hamiltonian. Note that M. Li [10] (also see [9]) considered the circumferences of these classes and proved that every 3-connected claw-free graph H of order n contains a cycle of length at least $\min\{6\delta(H) - 15, n\}$. Favaron and Fraisse [3] further improved these results in [11,7,8], and proved the following result. Note that Kuipers and Veldman [5] conjectured that every 3-connected claw-free graph H of order $n \leq 10\delta(H) - 6$ is hamiltonian if n is sufficiently large.

Theorem 1.3 (*Favaron and Fraisse [3]*)**.** *Let H be a 3-connected claw-free graph of order n. If $n \leq 10\delta(H) - 38$, then H is hamiltonian.*

Lai, Shao and Zhan [6] proved that the conjecture of Kuipers and Veldman [5] is true.

Theorem 1.4 (*Lai, Shao and Zhan [6]*)**.** *Let H be a 3-connected claw-free graph on $n \geq 196$ vertices. If $n \leq 10\delta(H) - 5$, then either H is hamiltonian, or $\delta(H) = (n + 5)/10$ and $cl(H)$ is the line graph of the graph G obtained from the Petersen graph $\mathcal{P}_{10}$ by adding $(n - 15)/10$ pendant edges at each vertex of $\mathcal{P}_{10}$.*

In this paper, our motivation is to improve the two results above. Let

$\mathcal{J}_1 = \{H$: H is a 3-connected non-hamiltonian claw-free graph such that its Ryjáček's closure $cl(H)$ is the line graph of one of the graphs obtained from the Petersen graph $\mathcal{P}_{10}$ by adding at least one pendant edge at each vertex of $\mathcal{P}_{10}$ and by subdividing m edges of $\mathcal{P}_{10}$ for $m = 0, 1, 2, \ldots, 15\}$, and

$\mathcal{J}_2 = \{H$: H is a 3-connected non-hamiltonian claw-free graph such that its Ryjáček's closure $cl(H)$ is the line graph of one of the graphs G obtained from the Petersen graph $\mathcal{P}_{10}$ by replacing exactly one vertex of $\mathcal{P}_{10}$ with $W = \bar{K}_{2,p}$ ($p \geq 2$) and by adding at least one pendant edge at all other nine vertices of $\mathcal{P}_{10}$, and by subdividing m edges of $\mathcal{P}_{10}$ for $m = 0, 1, 2, \ldots, 15$, where $\bar{K}_{2,p}$ is a bipartite graph, G is connected and $W = \bar{K}_{2,p}$ can be arbitrarily connected to $\mathcal{P}_{10}$ such that $1 \leq |N_W(V(G) - V(W))| \leq 3$, $|N_{V(G)-V(W)}(V(W))| = 3$, $|E_G(V(W), V(G) - V(W))| = 3$ and there are at most three vertices w of $d_G(w) = 2$ in $N_W(V(G) - V(W))\}$.

In this paper, we prove the following result, which is also the improvement of [11,7,9,8,10]. That is, the bounds $5\delta - 5$ [7], $6\delta - 9$ [11,8], $10\delta - 38$ [3], $10\delta - 6$ [6] are relaxed to $12\delta - 34$.

Theorem 1.5. *Let H be a 3-connected claw-free graph of order $n \geq 363$. If $n \leq 12\delta(H) - 34$, then either H is hamiltonian or $H \in \mathcal{J}_1 \cup \mathcal{J}_2$.*

Let

$\mathcal{J}_3 = \{H$: H is a 3-connected non-hamiltonian claw-free graph such that its Ryjáček's closure $cl(H)$ is the line graph of one of the graphs G obtained from the Petersen graph $\mathcal{P}_{10}$ by replacing exactly two vertices of $\mathcal{P}_{10}$ with $W = \bar{K}_{2,p}$ ($p \geq 2$) and by adding at least one pendant edge at all other eight vertices of $\mathcal{P}_{10}$, and by subdividing m edges of $\mathcal{P}_{10}$ for $m = 0, 1, 2, \ldots, 15$, where G is connected and $W = \bar{K}_{2,p}$ can be arbitrarily connected to $\mathcal{P}_{10}$ such that $1 \leq |N_W(V(G) - V(W))| \leq 3$, $|N_{V(G)-V(W)}(V(W))| = 3$, $|E_G(V(W), V(G) - V(W))| = 3$ and there are at most three vertices w of $d_G(w) = 2$ in $N_W(V(G) - V(W))\}$.