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Communication

Unimodality and log-concavity of f-vectors for cyclic and ordinary polytopes

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ABSTRACT

The question of unimodality of f-vectors of cyclic polytopes (which enumerate the number of faces of each dimension) is settled in the affirmative. More generally, the stronger property of log-concavity of f-vectors is seen to hold for the larger class of ordinary polytopes.

1. Introduction

In the 1950's Motzkin conjectured (see Björner [6]) that the f-vectors of convex polytopes are unimodal, i.e., that for every d-polytope there exists some j such that

$$f_{-1} \leq f_0 \leq \cdots \leq f_j \geq \cdots \geq f_{d-1},$$

where f_i is the number of i-dimensional faces of the polytope for $-1 \leq i \leq d-1$. We have $f_{-1} = 1$ for the improper face $\emptyset$ and f_{d-1} is the number of facets. The unimodality conjecture for convex polytopes was also stated by Welsh [15]. Danzer already showed in 1964 (see [18, Section 2.1]) that the conjecture cannot stand in its full generality, still leaving open the question: which natural classes of polytopes have unimodal f-vectors?

The unimodality conjecture fails even for simplicial polytopes (see Björner [6]) and even in low dimensions (for a counterexample see Eckhoff [9]). However, the conjecture holds for certain classes of polytopes with some restrictions on dimension (see e.g. Werner [16,17]). Unimodality also holds for some families of polytopes in any dimension. The classical example is the face vector of the d-simplex, which is identical to the dth row of Pascal's triangle.

The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is called *log-concave* if $f_{i-1}f_{i+1} \leq f_i^2$ for all $-1 < i < d-1$. The log-concavity of the f-vector of a polytope (being positive) implies its unimodality. The special shape of f-vectors of cyclic polytopes makes them applicable to certain constructions of polytopes with non-unimodal f-vector [18]. The question of whether the f-vectors of cyclic polytopes themselves are unimodal or not, has been open for some time. Eckhoff mentioned it in 2006 [9], and Ziegler reported in 2004 and 2007 [18, Section 2.2] that the challenge was still open except when the number of vertices is either small or very large with respect to d. More recently, additional partial results were also obtained by Schmitt [13].

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However, the methods used,

(a) direct computation,
(b) asymptotic analysis of the gap between n and d,
(c) attempted simplification of the intricate sum representing the f-vector,

did not lead to a complete answer to the question of unimodality. Our Theorem 2 settles this question in the affirmative by establishing log-concavity of f-vectors for the larger class of ordinary polytopes, introduced by Bisztriczky [4,5]. The argument is based on the description of the f-vectors of ordinary polytopes due to Dinh [8] and Bayer [1], involving also the use of a related vector (the h-vector) that was considered unhelpful in the context of earlier efforts [13]. A general combinatorial result of Brenti [7, Corollary 8.3] would also allow to transfer the property of log-concavity from h-vectors to f-vectors, but surprisingly this argument was not called upon to provide a general answer to the unimodality question for cyclic polytopes, even though previous discussions of the problem have recognized the relevance of Brenti's work for some time (Eckhoff [9], Schmitt [13]). The conclusion would not of course have been immediate without the use of the h-vector, nor would it apply to larger classes of polytopes whose h-vectors are not generally log-concave, such as the ordinary polytopes.

A first version of our unimodality proof for f-vectors of cyclic polytopes was contained in the manuscript [12].

2. *f*-vectors and *h*-vectors

For any d-polytope, its h-vector $(h_0, \dots, h_d)$ is defined by

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}$$

$$f_j = \sum_{i=0}^{d} \binom{d-i}{d-j-1} h_i, \quad \text{for } -1 \le j \le d-1.$$

These relations can be visualized following an observation of Stanley [14] as adapted by Lee in [11]. Let the first $d+1$ bordering 1's on the right-hand side of Pascal's triangle be replaced by the components of the h-vector, the internal entries (indicated by $\triangle$'s in the following figure) obeying the usual "sum of two entries above" rule: the f-vector emerges in the $(d+1)$th row of this modified Pascal's triangle.

$$\begin{array}{c}
\mathbf{h} \\
1 \quad \mathbf{v} \\
1 \quad \triangle \quad \mathbf{e} \\
1 \quad \triangle \quad \triangle \quad \mathbf{c} \\
1 \quad \triangle \quad \triangle \quad \triangle \quad \mathbf{t} \\
1 \quad \triangle \quad \triangle \quad \triangle \quad \triangle \quad \mathbf{o} \\
1 \quad \triangle \quad \triangle \quad \triangle \quad \triangle \quad \triangle \quad \mathbf{r} \\
\hline
\mathbf{f} \quad \mathbf{v} \quad \mathbf{e} \quad \mathbf{c} \quad \mathbf{t} \quad \mathbf{o} \quad \mathbf{r}
\end{array}$$

To handle this triangular array more formally we use the following operators N, F and T on vectors. The operator N from $\mathbb{R}^s \times \mathbb{R} = \mathbb{R}^{s+1}$ to $\mathbb{R}^{s+1}$ for any s is defined by

$$\begin{aligned} N(\mathbf{a}, b) &= N((a_{-1}, a_0, a_1, \dots, a_{s-2}), b) \\ &= (a_{-1}, a_{-1} + a_0, a_0 + a_1, \dots, a_{s-3} + a_{s-2}, a_{s-2} + b). \end{aligned}$$

The operator N produces a row of the modified Pascal's triangle from the previous row. For $\mathbf{b} = (b_0, b_1, \dots, b_r) \in \mathbb{R}^{r+1}$ define $F(\mathbf{b})$ as the $(r+1)$th (last) vector in the vector sequence

$$\mathbf{b}(1) = (b_0), \qquad \mathbf{b}(2) = N(\mathbf{b}(1), b_1), \dots, \qquad \mathbf{b}(i+1) = N(\mathbf{b}(i), b_i), \dots, \mathbf{b}(r+1).$$

In this notation, for the h-vector $\mathbf{h} = (h_0, h_1, \dots, h_d) \in \mathbb{R}^{d+1}$ of a d-polytope, Stanley's observation means that $F(\mathbf{h})$ is the f-vector of the polytope.

Finally for $\mathbf{a} = (a_{-1}, a_0, a_1, \dots, a_{s-2}) \in \mathbb{R}^s$ and $\mathbf{b} = (b_0, b_1, \dots, b_r) \in \mathbb{R}^{r+1}$, define $T(\mathbf{a}, \mathbf{b})$ as the $(r+1)$th (last) vector in the vector sequence

$$\mathbf{b}(1) = N(\mathbf{a}, b_0), \qquad \mathbf{b}(2) = N(\mathbf{b}(1), b_1), \dots, \qquad \mathbf{b}(i+1) = N(\mathbf{b}(i), b_i), \dots, \mathbf{b}(r+1).$$

For any $0 < i < d$ the f-vector of a d-polytope is

$$T\big(F(h_0, h_1, \dots, h_i), (h_{i+1}, \dots, h_d)\big). \tag{2.1}$$

The following lemma can be obtained by reformulating a special case of a general result of Kurtz [10] on triangular arrays, but it can also be easily proved directly by induction.

Lemma 1. *Let $\boldsymbol{a}$ be a log-concave vector and let $\boldsymbol{b} = (b_0, b_1, \dots, b_r)$ such that $b_0 \ge b_1 \ge \dots \ge b_r$. Then the vector $T(\boldsymbol{a}, \boldsymbol{b})$ is log-concave.*