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Some approaches for solving the general (t, k)-design existence problem and other related problems

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ABSTRACT

In this short survey, some approaches that can be used to solve the generalized (t, k)-design problem are considered. Special cases of the generalized (t, k)-design problem include well-known conjectures for t-designs, degree sequences of graphs and hypergraphs, and partial Steiner systems. Also described are some related problems such as the characterization of f-vectors of pure simplicial complexes, which are well known but little understood. Some suggestions how enumerative and polyhedral techniques may help are also described.

1. Introduction

Arrangements of objects into a collection of sets with specified incidence properties are of great importance in combinatorics and statistics. A special class of configurations is called *designs*. One of the most important types of design is balanced incomplete block design (BIBD).

Definition 1.1. Let X be a set of size v. A BIBD is an arrangement of elements of X into b sets (blocks) such that each set contains exactly k elements, each element is contained in exactly r different sets, and every $C \in \binom{X}{2}$ is contained in exactly λ sets.

BIBDs have many applications in the design of experiments in statistics. Hence much effort has been spent in developing methods for constructing BIBDs. There are many methods of constructions known. Most of the techniques for constructing BIBDs use results from finite groups, finite fields, and quadratic forms. Also many interesting composition techniques have been developed for the construction of BIBDs from smaller BIBDs. Bose was one of the first to give methodical constructions using groups and finite fields. Hanani solved the existence problem for BIBDs for all cases, when $k \leq 5$. Later Hanani, Bose, Shrikhande, Ray-Chaudhuri, Wilson, and several others improved these methods considerably. Finally, Wilson showed that for a given k and λ a BIBD exists, when v is sufficiently large. For more details, the reader is referred to [2,9]. But the general question, namely for what parameters a BIBD exists, has not yet been solved.

A natural generalization of a BIBD is a t-(v, k, λ) design.

Definition 1.2. Let X be a set of size v. A t-(v, k, λ) design is a collection of subsets of X of size k such that any element of $\binom{X}{t}$ is contained in exactly λ elements of this collection.

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The theory of t-designs is a well-developed field, and there are many interesting conjectures about these designs (see [2,7,32,33,23]).

Let $\mathcal{C}$ be a collection of subsets of X such that it is a t-(v, k, λ) design. Suppose that we want to count all possible pairs $((S, T), C)$ where S is any fixed subset of X of size i, T is a subset of X of size t, C is an element of $\mathcal{C}$, and $S \subseteq T \subseteq C$. We can do this count in two different ways. First we can count all t-subsets of X that contain S then count the number of times each t-subset appears in the collection $\mathcal{C}$. This implies the number of such pairs to be $\lambda \binom{v-i}{t-i}$. Another way would be counting all the t-subsets of C that contain S and then counting all the elements C of $\mathcal{C}$ that contain S. This implies the number to be $I \binom{k-i}{t-i}$, where I is an integer. Since these two numbers are equal, we have

$$\lambda \binom{v-i}{t-i} \equiv 0 \bmod \binom{k-i}{t-i}, \quad \text{for all } 0 \le i \le t.$$

Thus every t-(v, k, λ) design must satisfy the above equations.

A central problem in the theory of t-designs is the following.

Conjecture 1.3 (*Existence Conjecture*)**.** *Given $0 \le t < k \le v$, with v sufficiently large compared to k, a t-(v, k, λ) design exists if and only if*

$$\lambda \binom{v-i}{t-i} \equiv 0 \bmod \binom{k-i}{t-i}, \quad \textit{for all } 0 \le i \le t.$$

So far, the conjecture has been proved to be true for $t = 1$ and $t = 2$ only. In fact, most of the known results for t-designs are only for the case $t = 2$ (see [2]). For the case $t = 2$, as was earlier remarked, the conjecture in the above general form was proved by Wilson [33].

We now need to introduce some notation.

We will denote by $\mathbb{P}(X)$ the set all subsets of X, and by $\mathbb{P}_k(X)$ the set of all k-subsets of X, $0 \le k \le v$. We will denote by $V_k(X)$ the set of all rational-valued functions $f : \mathbb{P}_k(X) \to \mathbb{Q}$. Observe that $V_k(X)$ is a vector space over $\mathbb{Q}$, of dimension $\binom{v}{k}$. The set $M_k(X) \subseteq V_k(X)$ of all integral-valued functions is a module of rank $\binom{v}{k}$ over the ring of integers $\mathbb{Z}$.

Now, let $0 \le t \le k \le v$. For $f \in V_k(X)$, define $\partial_t f \in V_t(X)$ by $\partial_t f(T) = \Sigma f(B)$, where the sum is over all B satisfying $T \subseteq B$.

The function $j_k \in V_k(X)$ is defined by $j_k(B) = 1$ for all $B \in \mathbb{P}_k(X)$.

Let $N_{t,k} = N_{t,k}(X)$ denote the $\binom{v}{t} \times \binom{v}{k}$ matrix defined by

$$N_{t,k}(T, B) = \begin{cases} 1 & \text{if } T \subseteq B, \\ 0 & \text{otherwise,} \end{cases}$$

where $T \in \mathbb{P}_t(X)$ and $B \in \mathbb{P}_k(X)$.

Now, let $D = (X, f)$, where $f \in V_k(X)$. D is called a *rational* t-(v, k, λ) design if $\partial_t f = \lambda j_t$. Such a pair is called a *signed* t-(v, k, λ) design if f is integral, and a t-(v, k, λ) design if f is integral and non-negative ($f \ge 0$), i.e., for all $B \in \mathbb{P}_k(X)$, $f(B) \ge 0$ and $f(B) \in \mathbb{Z}$. The number $f(B)$ may be thought of as the frequency with which the k-subset B occurs in the design. When $f(B)$ is non-zero, the k-subset B is called a block of the design.

For a collection $\mathcal{C}$ of subsets of size k, consider its frequency function $f \in M_k(X)$ to be the function defined by $f(B) =$ the number of times B occurs in the collection $\mathcal{C}$. We note that $\partial_t f = \lambda j_t$ if and only if $\mathcal{C}$ is a t-(v, k, λ) design. Thus, the two definitions of a t-(v, k, λ)-design we have given are essentially the same.

We will fix X throughout this paper. Then we may refer to f itself as a t-design if it satisfies above conditions. For the usual definition of t-designs as a family of subsets, f corresponds to the frequency vector of occurrence of k-subsets in the family.

Another well-known problem concerning families of sets is the following characterization problem for degree sequences of a uniform hypergraph.

Problem 1.4 (*Characterization Problem*)**.** Let $h \in V_t(X)$, $t = 1$. Give necessary and sufficient conditions, which can be verified algorithmically (in polynomial time), for the existence of an $f \in V_k(X)$ such that $f(B) \in \{0, 1\}$ for all $B \in \mathbb{P}_k(X)$ and $\partial_1 f = h$.

Note that, for $k = 2$, the problem corresponds to characterizing the degree sequence of the graphs. The well-known Havel–Hakimi theorem and the Erdős–Gallai theorem are such characterizations. Also, for general k, if we allow multiple edges, that is we replace the condition $f(B) \in \{0, 1\}$ by the condition $f(B)$ is a non-negative integer, then a good characterization is provided by the well-known Gale–Ryser theorem [15,25]. These are essentially the only general cases for which a good characterization is known.

We can now state a general (t, k)-existence problem, special cases of which include many well-known problems in combinatorics, apart from the above problems. Some examples are existence problems on partial Steiner systems and