Discrete Applied Mathematics 157 (2009) 685–690

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimally 3-restricted edge connected graphs☆

Qinghai Liu, Yanmei Hong, Zhao Zhang *

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China

ARTICLE INFO

Article history:
Received 12 December 2007
Received in revised form 13 June 2008
Accepted 31 July 2008
Available online 27 August 2008

Keywords:
Fault tolerance
Restricted edge connectivity

ABSTRACT

For a connected graph $G = (V, E)$, an edge set $S \subset E$ is a 3-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has order at least three. The cardinality of a minimum 3-restricted edge cut of G is the 3-restricted edge connectivity of G, denoted by $\lambda_3(G)$. A graph G is called minimally 3-restricted edge connected if $\lambda_3(G - e) < \lambda_3(G)$ for each edge $e \in E$. A graph G is λ_3-optimal if $\lambda_3(G) = \xi_3(G)$, where $\xi_3(G) = \max\{\omega(U) : U \subset V(G),\ G[U]$ is connected, $|U| = 3\}$, $\omega(U)$ is the number of edges between U and $V \setminus U$, and $G[U]$ is the subgraph of G induced by vertex set U. We show in this paper that a minimally 3-restricted edge connected graph is always λ_3-optimal except the 3-cube.

1. Introduction

A network can be conveniently modeled as a graph $G = (V, E)$. A classic measure of the fault tolerance of a network is the edge connectivity $\lambda(G)$. In general, the larger $\lambda(G)$ is, the more reliable the network is [2]. A more refined measure known as restricted edge connectivity was proposed by Esfahanian and Hakimi [5], which was further generalized to k-restricted edge connectivity by Fábrega and Fiol [6] (called k-extra edge connectivity in their paper).

Let G be a connected graph. An edge set $S \subset E(G)$ is said to be a *k-restricted edge cut* of G if $G - S$ is disconnected and each component of $G - S$ has at least k vertices. The minimum cardinality of a k-restricted edge cut is called the *k-restricted edge connectivity of* G, denoting by $\lambda_k(G)$. A k-restricted edge cut S with $|S| = \lambda_k(G)$ is called a *λ_k-cut*. Not all graphs have λ_k-cuts [4,5,12,21]. Those which do have λ_k-cuts are called *λ_k-connected* graphs. According to current studies on k-restricted edge connectivity [10,11,13,17], it seems that the larger $\lambda_k(G)$ is, the more reliable the network is. In [21], Zhang and Yuan proved that $\lambda_k(G) \leqslant \xi_k(G)$ holds for any integer $k \leqslant \delta(G) + 1$ except for a class of graphs (such a graph is constructed from a set of complete subgraphs K_δ by adding a new vertex u and connect u to every other vertex), where $\delta(G)$ is the minimum degree of G and $\xi_k(G) = \min\{\omega(U) : U \subset V(G)$, G[U] is connected,$|U| = k\}$, $\omega(U)$ is the number of edges between U and $V \setminus U$, and $G[U]$ is the subgraph of G induced by U. A graph G is called *λ_k-optimal* if $\lambda_k(G) = \xi_k(G)$. There is much research on sufficient conditions for a graph to be λ_k-optimal, such as symmetric conditions [11,13,17,18], degree conditions [14,15,19], and girth-diameter conditions [1,6,16,20]. For more information on this topic, we refer the readers to the nice survey paper by Hellwig and Volkmann [7].

In this paper, we give another type of sufficient condition called a minimally restricted edge connected condition. A graph G is a *minimally k-restricted edge connected graph* (*minimally λ_k-graph* for short) if $\lambda_k(G - e) < \lambda_k(G)$ (and thus $\lambda_k(G - e) = \lambda_k(G) - 1$) for each edge $e \in E(G)$. It is implied in the definition that $\lambda_k(G - e)$ exists for each edge e. If e is a pending edge, then $G - e$ does not have λ_k-cut for $k \geqslant 2$. So, we always assume $\delta(G) \geqslant 2$ when G is a minimally λ_k-graph for some $k \geqslant 2$. A minimally λ_1-graph is exactly a minimally edge connected graph, which has been shown to be λ-optimal ([9] Exercise 49). In [8], the authors have proved that every minimally λ_2-graph is λ_2-optimal. In this paper, we show that every minimally λ_3-graph is always λ_3-optimal except the 3-cube.

☆ This research is supported by NSFC (60603003), the Key Project of Chinese Ministry of Education (208161) and XJEDU.
* Corresponding author. Tel.: +86 13899960204; fax: +86 991 8585505.
E-mail address: zhzhao@xju.edu.cn (Z. Zhang).

0166-218X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2008.07.009

2. Preliminaries and terminologies

Let $G = (V, E)$ be a graph. For two disjoint vertex sets $U_1, U_2 \subset V(G)$, denote by $[U_1, U_2]_G$ the set of edges of G with one end in U_1 and the other end in U_2, $G[U]$ is the subgraph of G induced by vertex set $U \subset V(G)$, $\overline{U} = V(G) \setminus U$ is the complement of U, $\omega_G(U) = |[U, \overline{U}]_G|$ is the number of edges between U and $\overline{U}$. When the graph under consideration is obvious, we omit the subscript G. Write $d_A(U) = |[U, A \setminus U]|$, $d_A(u) = d_A(\{u\})$. Sometimes, we use a graph itself to represent its vertex set. For example, $\omega(C)$ is used instead of $\omega(V(C))$, where C is a subgraph of G; for an edge $e = uv$, $d_A(e)$ is used instead of $d_A(\{u, v\})$, etc.

A λ_3-*fragment* is a subset U of $V(G)$ with $[U, \overline{U}]$ being a λ_3-cut. If U is a λ_3-fragment, then so is $\overline{U}$, and both $G[U]$ and $G[\overline{U}]$ are connected. A λ_3-fragment with minimum order is called a λ_3-*atom*. The order of a λ_3-atom is denoted by $\alpha_3(G)$. Clearly, $\alpha_3(G) \leqslant |V(G)|/2$.

A graph H is λ_3-*independent* if each component of H has at most two vertices. A connected graph of order at most two is λ_3-*trivial*. A graph is called λ_3-*non-trivial* if it has a component which contains at least three vertices.

The following two observations will be used frequently without mentioning them explicitly. The first is that if two connected subgraphs G_1 and G_2 have nonempty intersection, then $G_1 \cup G_2$ is also connected. The second is that for a vertex set F of a connected graph G and a component C of $G - F$, if $G[F]$ is connected, then so is $G - C$.

For terminologies not given here, we refer to [3] for reference.

3. Main result

First, it should be noted that if $\alpha_3(G) = 3$, then G is λ_3-optimal. In fact, Bonsma et al. [4] have proved that $\lambda_3(G) \leqslant \xi_3(G)$ holds for any λ_3-connected graph G. On the other hand, considering a λ_3-atom A of G, we have $\lambda_3(G) = \omega(A) \geqslant \xi_3(G)$. So, $\lambda_3(G) = \xi_3(G)$. In view of this observation, to derive our main theorem, it suffices to show that $\alpha_3(G) = 3$. In the following, we prove that if G is a minimally 3-restricted edge connected graph with $\alpha_3(G) \geqslant 4$, then G is isomorphic to 3-cube.

Lemma 1. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$, F be a subset of G with $G[\overline{F}]$ being connected. If one of the following conditions is satisfied:*

(a) $\omega(F) < \lambda_3(G)$, *or*
(b) $\omega(F) = \lambda_3(G)$ *and* $|F| < \alpha_3(G)$,

then $G[F]$ is λ_3-independent.

Proof. Suppose F has a λ_3-non-trivial component C. Since $G[\overline{C}]$ is connected, we have $\omega(F) \geqslant \omega(C) \geqslant \lambda_3(G)$, which is clearly a contradiction to condition (a). If condition (b) occurs, then $\omega(F) = \omega(C) = \lambda_3(G)$. Hence $V(C)$ is a λ_3-fragment. But then $|F| \geqslant |C| \geqslant \alpha_3(G)$, contradicting $|F| < \alpha_3(G)$. □

Lemma 2. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$ and $\alpha_3(G) \geqslant 4$, A be a λ_3-atom of G, and B be a λ_3-fragment of G.*

(a) *If a subset U of A is such that $G[U]$ is connected and $G[A \setminus U]$ has a λ_3-non-trivial component, then $d_A(U) > d_{\overline{A}}(U)$.*
(b) *If a subset U of B is such that $G[U]$ is connected and $G[B \setminus U]$ has a λ_3-non-trivial component, then $d_B(U) \geqslant d_{\overline{B}}(U)$.*
(c) $\delta(G[A]) \geqslant 2$.

Proof. (a). Suppose $d_A(U) \leqslant d_{\overline{A}}(U)$. Then $\omega(A \setminus U) = \omega(A) + d_A(U) - d_{\overline{A}}(U) \leqslant \omega(A) = \lambda_3(G)$. By noting that $|A \setminus U| < |A| = \alpha_3(G)$, it follows from Lemma 1 that $G[A \setminus U]$ is λ_3-independent, a contradiction.
(b). The proof of (b) is similar to that of (a); note that under the assumption $d_B(U) < d_{\overline{B}}(U)$, it can be deduced that $\omega(B \setminus U) < \lambda_3(G)$.
(c) is a consequence of (a). In fact, for each vertex $x \in A$, if $A - x$ is λ_3-independent, then $d_A(x) \geqslant 2$ since $|A| \geqslant 4$. If $A - x$ contains a λ_3-non-trivial component, taking $U = \{x\}$ in (a), we have $d_A(x) > d_{\overline{A}}(x)$. Then it follows from $d_A(x) > \frac{1}{2} \cdot (d_A(x) + d_{\overline{A}}(x)) = \frac{1}{2} \cdot d_G(x) \geqslant 1$ that $d_A(x) \geqslant 2$. □

Similar to Lemma 2, we can prove the following lemma. The key observation to the proof, as well as some proofs after it, is that for any edge $e \in E(G)$, if $\lambda_3(G - e) < \lambda_3(G)$, then any λ_3-fragment of $G - e$ contains exactly one end of e, and is a λ_3-fragment of G. Note that the observation is true when G is minimally 3-restricted edge connected.

Lemma 3. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$ and $\alpha_3(G) \geqslant 4$, $e = uv$ be an edge of G, $\lambda_3(G - e) < \lambda_3(G)$, A be a λ_3-atom of $G - e$ with $u \in A$ and $v \notin A$.*

(a) *If a subset U of A is such that $G[U]$ is connected, $G[A] - U$ has a λ_3-non-trivial component, and e is not incident with U, then $d_A(U) > d_{\overline{A}}(U)$.*
(b) $d_{G[A]}(x) \geqslant 2$ *for each $x (\neq u) \in A$.*

Lemma 4. *Let G be a λ_3-connected graph with $\delta(G) \geqslant 2$ and $\alpha_3(G) \geqslant 4$, A be a λ_3-atom of G, B be a λ_3-fragment of G, $A \cap B \neq \emptyset$. Then for any component C of $G[A \cap B]$, either $G[A] - C$ or $G[B] - C$ is λ_3-independent.*