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On dominating sets of maximal outerplanar graphs[☆]

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ABSTRACT

A dominating set of a graph is a set S of vertices such that every vertex in the graph is either in S or is adjacent to a vertex in S. The domination number of a graph G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. We show that if G is an n-vertex maximal outerplanar graph, then $\gamma(G) \leq (n+t)/4$, where t is the number of vertices of degree 2 in G. We show that this bound is tight for all $t \geq 2$. Upper-bounds for $\gamma(G)$ are known for a few classes of graphs.

1. Introduction

In this paper we only consider finite, undirected and simple graphs. A *dominating set* of a graph $G = (V, E)$ is a set $S \subseteq V$ such that every vertex in G is either in S or is adjacent to a vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. Garey and Johnson [1] showed that deciding whether a given graph has domination number at most some given integer k is an NP-complete problem; and remains so for planar graphs with maximum degree 3 and planar 4-regular graphs.

A graph G is *outerplanar* if it has an embedding in the plane such that all vertices belong to the boundary of its outer face (the unbounded face). An outerplanar graph G is *maximal* if $G + uv$ is not outerplanar for any two non-adjacent vertices u and v of G. In this paper we are concerned with upper bounds for $\gamma(G)$, when G is a maximal outerplanar graph.

In 1996, Matheson and Tarjan [8] proved a tight upper bound for the dominating number on the class of *triangulated discs*: graphs that have an embedding in the plane such that all of their faces are triangles, except possibly one. They proved that $\gamma(G) \leq n/3$ for any n-vertex triangulated disc. They also showed that this bound is tight. Plummer and Zha [10] extended this bound to triangulations on the projective plane and proved that $\gamma(G) \leq \lceil n/3 \rceil$ for triangulations on the torus or Klein bottle. They also showed that this bound is tight. (A triangulated disc in which all faces are triangles and any two face boundaries intersect in a single edge, a single vertex, or not at all is called a triangulation.) Honjo et al. [5] extended the latter results showing the bound $n/3$ for triangulations on the torus and the Klein bottle and also for some other surfaces.

Matheson and Tarjan conjectured that $\gamma(G) \leq n/4$ for every n-vertex plane triangulation G with n sufficiently large. They noted that the octahedron, which has 6 vertices, has domination number 2. Recently, King and Pelsmajer [6] proved that this conjecture is true for graphs with maximum degree at most 6.

We observed that the graphs given by Matheson and Tarjan to show that the upper-bound $n/3$ is tight for triangulated discs are, in fact, outerplanar graphs. So, we came naturally to the question of whether this bound would also be the best

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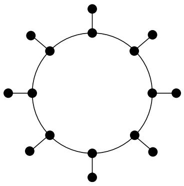

Fig. 1. An outerplanar graph with domination number $n/2$.

possible for *maximal outerplanar* graphs. For outerplanar graphs that are not maximal, the upper-bound $n/2$ is the best possible, as Fig. 1 shows. We prove a simple upper-bound for $\gamma(G)$ when G is a maximal outerplanar graph, and show that this bound is tight.

2. Basic results

We first observe that if G is a maximal outerplanar graph, then G is 2-connected and Hamiltonian. It is also immediate that the following holds.

Proposition 1. *If G is a maximal outerplanar graph, then there is an embedding of G in the plane such that the boundary of the outer face is a Hamiltonian cycle and each inner face is a triangle.* □

A maximal outerplanar graph embedded in the plane as mentioned above will be called a *maximal outerplane graph*. For such a graph G, we denote by H_G the (unique) Hamiltonian cycle which is the boundary of the outer face. We omit the subscript G, when G is clear from the context.

Let f be an inner face of a maximal outerplane graph G. If f is adjacent to the outer face, then we say that f is a *marginal triangle*; otherwise we say that f is an *internal triangle*. A maximal outerplane graph G without internal triangles is called *striped*. We may use the term *triangle* to refer to an inner face or to a subgraph that is isomorphic to K_3.

It is interesting to note the direct relation between the number of internal triangles and the number of vertices of degree 2 in a maximal outerplane graph. This is stated in the next proposition.

Proposition 2. *Let G be a maximal outerplane graph of order $n \geq 4$. If G has k internal triangles, then G has $k+2$ vertices of degree 2.*

Proof. The proof can be done by induction on k. For $k = 0$ the proof follows easily by induction on n (one can contract an edge of H_G that belongs to a marginal triangle which has two chords of H_G). □

The next result will be useful in what follows.

Lemma 3. *If G is a maximal outerplane graph of order $n \geq 3$, then G has $n-1$ faces and H_G has $n-3$ chords.*

Proof. First note that if G has n_f faces and H_G has n_c chords, then G has $n+n_c$ edges, and thus, by Euler's formula, $n_f = n_c+2$. The dual graph G^* of G has $n_f - 1$ vertices of degree 3 and one vertex of degree n. Adding the degrees of the vertices of G^*, we get that $3(n_f - 1) + n = 2(n + n_c)$. The two equations yield $n_f = n - 1$ and $n_c = n - 3$. □

3. Maximal outerplanar graphs without internal triangles

In this section we prove an upper-bound for the domination number of striped maximal outerplanar graphs. For that, we introduce first a terminology and notation that will be useful in this proof.

Let G be a striped maximal outerplane graph of order $n \geq 4$, and let $H = (x_0, x_1, \ldots, x_{n-1})$ be the boundary of its outer face. From Proposition 2, we know that G has exactly two vertices of degree 2. Suppose, without loss of generality, that x_0 and x_{k+1} are the two vertices of degree 2 in G.

The removal of the vertices x_0 and x_{k+1} breaks the cycle H into two paths: the path $P = (x_1, x_2, \ldots, x_k)$ and the path $Q = (x_{k+2}, x_{k+3}, \ldots, x_{n-1})$. To refer to the reverse of Q, for ease of notation, we label its vertices in such a way that $Q^{-1} = (y_1, y_2, \ldots, y_t)$, as depicted in Fig. 2(a). We call this label assignment a *canonical vertex-labelling*.

We also consider that the chords of H are ordered and named $c_1, c_2, \ldots, c_{n-3}$, according to the statement of the next lemma (see Fig. 2(b)).

Lemma 4. *Let G be a striped maximal outerplane graph of order $n \geq 4$ endowed with a canonical vertex-labelling. Then, the chords of H can be ordered from c_1 to c_{n-3} in such a way that:*

(i) *$c_1 := x_1y_1$ and*
(ii) *if $c_p = x_iy_j$, then $c_{p+1} = x_iy_{j+1}$ or $c_{p+1} = x_{i+1}y_j$.* □