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The Ramsey number for a cycle of length six versus a clique of order eight

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ABSTRACT

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or the complement of G contains G_2. Let C_m denote a cycle of length m and K_n a complete graph of order n. In this paper, it is shown that $R(C_6, K_8) = 36$.

1. Introduction

All graphs considered in this paper are finite simple graph without loops. For two given graphs G_1 and G_2, the *Ramsey number* $R(G_1, G_2)$ is the smallest integer n such that for any graph G of order n, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *minimum degree* of G is denoted by $\delta(G)$. Define $\sigma_2(G) = \min\{d(u) + d(v) \mid u, v \in V(G) \text{ and } uv \notin E(G)\}$. Let $V_1, V_2 \subseteq V(G)$. We use $E(V_1, V_2)$ to denote the set of the edges between V_1 and V_2. The *independence number* of a graph G is denoted by $\alpha(G)$. For $U \subseteq V(G)$, we write $\alpha(U)$ for $\alpha(G[U])$, where $G[U]$ is the subgraph induced by U in G. A cycle and a path of order n are denoted by C_n and P_n, respectively. A *clique* or *complete graph* of order n is denoted by K_n. We use mK_n to denote the union of m vertex disjoint K_n. A *Wheel* of order $n + 1$ is $W_n = K_1 + C_n$ and W_n^- is a graph obtained from W_n by deleting a spoke from W_n. A *Book* $B_n = K_2 + \overline{K_n}$ is a graph of order $n + 2$. For notations not defined here, we follow [2].

The cycle-complete graph Ramsey number $R(C_m, K_n)$ was first studied by Erdös et al. in [5]. In the paper, they asked the following.

Question 1 (*Erdös et al. [5]*). For a given n, what is the smallest value of m such that $R(C_m, K_n) = (m - 1)(n - 1) + 1$?

Furthermore, they posed the following.

Conjecture 1 (*Erdös et al. [5]*). *$R(C_m, K_n) = (m - 1)(n - 1) + 1$ for $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$.*

The conjecture was confirmed for $n = 3$ in early works on Ramsey theory [6,9]. Yang et al. [11] proved the conjecture for $n = 4$.

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Table 1
Known Ramsey numbers $R(C_m, K_n)$ for $m \le n - 1$

	K_4	K_5	K_6	K_7	K_8	K_9
C_3	9	14	18	23	28	36
C_4		14	18	22	26	
C_5			21	25		
C_6				31		
C_7					43	

Theorem 1 (*Yang et al. [11]*). *$R(C_m, K_4) = 3m - 2$ for $m \ge 4$.*

Bollobás et al. [1] showed that the conjecture is true for $n = 5$.

Theorem 2 (*Bollobás et al. [1]*). *$R(C_m, K_5) = 4m - 3$ for $m \ge 5$.*

Schiermeyer [10] confirmed the conjecture for $n = 6$.

Theorem 3 (*Schiermeyer [10]*). *$R(C_m, K_6) = 5m - 4$ for $m \ge 6$.*

Chen et al. [3] proved the conjecture for the case when $n = 7$.

Theorem 4 (*Chen et al. [3]*). *$R(C_m, K_7) = 6m - 5$ for $m \ge 7$.*

So far, the conjecture is still open. All the results obtained indicate that the conjecture is true. However, it seems hard to understand the behavior of the Ramsey number $R(C_m, K_n)$ for the case when $m \le n - 1$. One reason for this may be that $R(C_3, K_n)$ is the classical Ramsey number $R(3, n)$ and the classical Ramsey numbers are very hard to determine. By now, only 14 exact values of $R(C_m, K_n)$ for $m \le n - 1$, including 6 classical Ramsey numbers, are known, see Table 1. All the details in Table 1 can be found in the dynamic survey [8].

In this paper, we calculate the value of the Ramsey number $R(C_6, K_8)$. The main result is the following.

Theorem 1. $R(C_6, K_8) = 36$.

Remark. Let $f(n)$ be the smallest value of m such that $R(C_m, K_n) = (m - 1)(n - 1) + 1$ for a given n. By the known results (see [8]), we have $f(3) = 4, f(4) = 4, f(5) = 5, f(6) = 5$ and $f(7) = 5$. Obviously, Theorem 1 shows that $f(8) \le 6$. It is not known by now whether $f(8) = 5$ holds. In general, we have the following.

Question 2. Does there exist a constant N such that $f(n) \le N$? Furthermore, is it true that $f(n) = 5$ for $n \ge 5$?

2. Preliminaries

Lemma 1 (*McKay and Zhang [7]*). $R(K_3, K_8) = 28$.

Lemma 2 (*Cheng et al. [4]*). $R(C_6, K_7) = 31$.

Lemma 3. *Let G be a graph of order 36 with $\alpha(G) \le 7$. If G contains no C_6, then $\delta(G) \ge 5$.*

Proof. If there is some vertex v such that $d(v) \le 4$, then $G' = G - N[v]$ is a graph of order at least 31. By Lemma 2, $\alpha(G') \ge 7$. Thus, an independent set of order at least 7 in G' together with v form an independent set of order at least 8 in G, which contradicts $\alpha(G) \le 7$. ■

Lemma 4. *Let G be a graph of order 36 with $\alpha(G) \le 7$. If G contains no C_6, then G contains no K_5.*

Proof. Suppose to the contrary that $K_5 = H \subset G$ with $V(H) = \{w_i \mid 0 \le i \le 4\}$. Set $U = V(G) - V(H)$. By Lemma 3, $\delta(G) \ge 5$. Thus, we have $N_U(w_i) \ne \emptyset$ for $0 \le i \le 4$. Let $v_i \in N_U(w_i)$ and $V_i = N_U[v_i]$, where $0 \le i \le 4$. Since G contains no C_6, we have

$$N(V_i) \cap V(H) = \{w_i\} \quad \text{for } 0 \le i \le 4, \tag{1}$$

$$V_i \cap V_j = \emptyset \quad \text{for } 0 \le i < j \le 4, \tag{2}$$

and

$$E(V_i, V_j) = \emptyset \quad \text{for } 0 \le i < j \le 4. \tag{3}$$

By (1), we have $d_H(v_i) = 1$, which implies $|V_i| \ge 5$ for $0 \le i \le 4$ since $\delta(G) \ge 5$. Assume without loss of generality that $\alpha(V_0) \ge \alpha(V_1) \ge \alpha(V_2) \ge \alpha(V_3) \ge \alpha(V_4)$.