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Maximum weight independent sets in hole- and dart-free graphs

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ABSTRACT

The *Maximum Weight Independent Set (MWIS)* problem on graphs with vertex weights asks for a set of pairwise nonadjacent vertices of maximum total weight. The complexity of the MWIS problem for hole-free graphs is unknown. In this paper, we first prove that the MWIS problem for (hole, dart, gem)-free graphs can be solved in $O(n^3)$-time. By using this result, we prove that the MWIS problem for (hole, dart)-free graphs can be solved in $O(n^4)$-time. Though the MWIS problem for (hole, dart, gem)-free graphs is used as a subroutine, we also give the best known time bound for the solvability of the MWIS problem in (hole, dart, gem)-free graphs.

1. Introduction

In an undirected graph G, an *independent set* is a set of mutually nonadjacent vertices in G. The *Maximum Weight Independent Set (MWIS)* problem asks for an independent set of total maximum weight in the given graph G with vertex weight function w on $V(G)$. The MWIS problem is one of the fundamental algorithmic graph problems which frequently occurs as a subproblem in models in computer science and operations research. The MWIS problem is known to be *NP*-hard in general, and remains *NP*-hard even on restricted classes of graphs such as triangle-free graphs [16], and ($K_{1,4}$, diamond)-free graphs [10]. On the other hand, the MWIS problem is known to be solvable in polynomial time on many graph classes which are subclasses of P_5-free graphs [7,5,12]. In this context, as a generalization of trees and chordal graphs, hole-free graphs have been studied recently [1–3,6] because the cycle properties of graphs and their algorithmic aspects play a crucial role in combinatorial optimization, discrete mathematics and computer science. Chordal graphs, weakly chordal graphs, and perfect graphs are characterized in terms of cycle properties; and these classes of graphs are fundamental for algorithmic graph theory and various applications. Also, note that the class of (P_5, C_5)-free graphs is a subclass of hole-free graphs.

While the complexity of the MWIS problem for hole-free graphs is unknown, it is shown to be solvable in polynomial time for some subclasses of graphs, namely, (hole, diamond)-free graphs [1], (hole, co-chair)-free graphs [2], (hole, paraglider)-free graphs [3], and (hole, apple)-free graphs [6]; see Fig. 1. In this paper, we are concerned with the class of (hole, dart)-free graphs. The class of dart-free graphs includes certain well studied classes of graphs such as claw-free graphs, diamond-free graphs, and paw-free graphs.

An algorithm for finding a maximum weight independent set in a given weighted graph via clique separator decomposition has been given by Tarjan [19] (see also [21]). Using this technique, it has been proved that the MWIS problem for (hole, co-chair)-free graphs can be solved in $O(n^7)$-time [2], and it can be solved in $O(n^4)$-time for (hole, paraglider)-free graphs [3]. We use the same framework to prove that the MWIS problem for (hole, dart)-free graphs can be solved in $O(n^4)$-time. To do this, we consider a subclass of (hole, dart)-free graphs, namely, (hole, dart, gem)-free graphs, and we prove

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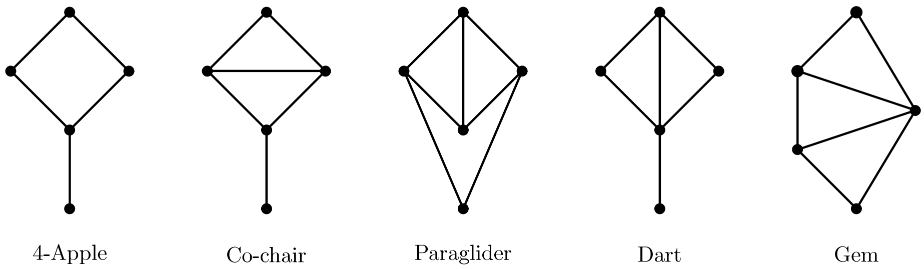

Fig. 1. Some special graphs.

that the MWIS problem for these classes of graphs can be solved in $O(n^3)$-time. Next, we use this result to solve the MWIS problem for (hole, dart)-free graphs. Though the MWIS problem for (hole, dart, gem)-free graphs is used as a subroutine, we also give the best known time bound for the solvability of the MWIS problem in (hole, dart, gem)-free graphs, since the only other known way to do it is by using the fact that (hole, dart, gem)-free graphs are perfect (by the Strong perfect graph theorem [8]), and the MWIS problem on perfect graphs can be solved in polynomial time [13].

Let $\mathcal{G}$ be a hereditary class of graphs. As regards the computational complexity of the application of Tarjan's method [19], initially in [5,6], it was assumed that if the MWIS problem can be solved in time $T = O(f(n))$ for the atoms in $\mathcal{G}$, then the MWIS problem can be solved in time $O(mn + n^2 \cdot T)$ for graphs in $\mathcal{G}$. Later, Brandstädt et al. [3] observed that if the MWIS problem can be solved in time $T = O(f(n))$ for the atoms in $\mathcal{G}$, then the MWIS problem can be solved in time $O(mn + n \cdot T)$ for graphs in $\mathcal{G}$.

Suppose that the atoms of $\mathcal{G}$ are nearly $\mathcal{C}$, where $\mathcal{C}$ is a hereditary property. Then in [4] it was observed that if the MWIS problem can be solved in time $T = O(f(n))$ for graphs with property $\mathcal{C}$, then the MWIS problem can be solved in time $O(n^2 \cdot T)$ for graphs in $\mathcal{G}$. In this paper, we show that if the MWIS problem can be solved in time $T = O(f(n))$ for graphs with property $\mathcal{C}$, then the MWIS problem is solvable in time $O(n \cdot T)$ for graphs in $\mathcal{G}$. Using this result, we observe that the time complexity of the solvability of the MWIS problem for (P_5, paraglider)-free graphs, and (P_6, C_4)-free graphs given in [4] can be reduced. We also give an analysis when the atoms are of specific graph classes where the MWIS problem can be solved efficiently in polynomial time. Using this result, we observe that the MWIS problem for (hole, paraglider)-free graphs can be solved in $O(n^3)$-time which is an improvement over the result given in [3].

2. Notation and terminology

For notation and terminology not defined here, we follow West [20]. Throughout this paper, let $G(V, E)$ be a finite, undirected and simple graph with $|V(G)| = n$ and $|E(G)| = m$. We also denote the vertex set of G as $V(G)$, and the edge set of G as $E(G)$. If $S \subseteq V(G)$, then $[S]$ denotes the subgraph induced by S in G. If $\{v_1, v_2, \ldots, v_k\} \subseteq V(G)$, then we simply write $[v_1, v_2, \ldots, v_k]$ instead of $[\{v_1, v_2, \ldots, v_k\}]$. If H is an induced subgraph of G, we write $H \sqsubseteq G$. For a graph G, let $\overline{G}$ denote the *complement graph* of G. A graph G is a *vertex-weighted* graph if each vertex of G is assigned a positive integer, the *weight* of the vertex.

The symbols P_n and C_n respectively denote the chordless path and chordless cycle on n vertices. The *length* of the path is the number of vertices in it. A *hole* is a C_k with $k \geq 5$. If $\mathcal{F}$ is a family of graphs, G is said to be $\mathcal{F}$*-free* if it contains no induced subgraph isomorphic to any graph in $\mathcal{F}$. A graph G is *chordal* if it is hole-free and C_4-free.

For $v \in V(G)$, the *neighborhood* $N(v)$ of v is the set $\{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. The neighborhood $N(X)$ of a subset $X \subseteq V(G)$ is the set $\{u \in V(G) \mid u \text{ is adjacent to a vertex of } X\}$. Given a subgraph H of G and $v \in V(G) \setminus V(H)$, let $N_H(v)$ denote the set $N(v) \cap V(H)$, and for $X \subseteq V(G) \setminus V(H)$, let $N_H(X)$ denote the set $N(X) \cap V(H)$.

Let $\mathcal{C}$ denote a (hereditary) graph property. A graph G is *nearly* $\mathcal{C}$ if for every $v \in V(G)$, $[V(G)\, N[v]]$ has the property $\mathcal{C}$.

An independent set (clique) in G is a subset of pairwise nonadjacent (adjacent) vertices in G. If G is a vertex weighted graph with vertex weight function w, then the maximum total weight of an independent set in G is called the *weighted independence number* of G, and is denoted by $\alpha_w(G)$. Obviously, the following identity holds:

$$\alpha_w(G) = \max_{v \in V(G)} \{w(v) + \alpha_w([\overline{N}(v)])\}. \tag{1}$$

Thus, whenever the MWIS problem can be solved in time T on a graph class with property $\mathcal{C}$, then it can be solved in time $(n \cdot T)$ for graphs having the nearly $\mathcal{C}$ property. For example, Grötschel et al. [13] gave a polynomial time algorithm for the MWIS problem on perfect graphs. Thus, the MWIS problem can be solved in polynomial time for nearly perfect graphs.

A subset $M \subseteq V(G)$ is a *module* in G if every vertex $v \in V(G) \setminus M$ is adjacent either to all of the vertices in M or to none of the vertices in M. A graph G is *prime* if it has no module M with $1 < |M| < |V(G)|$.