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Characterization of removable elements with respect to having k disjoint bases in a matroid

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ABSTRACT

The well-known spanning tree packing theorem of Nash-Williams and Tutte characterizes graphs with k edge-disjoint spanning trees. Edmonds generalizes this theorem to matroids with k disjoint bases. For any graph G that may not have k-edge-disjoint spanning trees, the problem of determining what edges should be added to G so that the resulting graph has k edge-disjoint spanning trees has been studied by Haas (2002) [11] and Liu et al. (2009) [17], among others. This paper aims to determine, for a matroid M that has k disjoint bases, the set $E_k(M)$ of elements in M such that for any $e \in E_k(M)$, $M - e$ also has k disjoint bases. Using the matroid strength defined by Catlin et al. (1992) [4], we present a characterization of $E_k(M)$ in terms of the strength of M. Consequently, this yields a characterization of edge sets $E_k(G)$ in a graph G with at least k edge-disjoint spanning trees such that $\forall e \in E_k(G)$, $G - e$ also has k edge-disjoint spanning trees. Polynomial algorithms are also discussed for identifying the set $E_k(M)$ in a matroid M, or the edge subset $E_k(G)$ for a connected graph G.

1. Introduction

The number of edge-disjoint spanning trees in a network, when modeled as a graph, often represents certain strength of the network [8]. The well-known spanning tree packing theorem of Nash-Williams [18] and Tutte [23] characterizes graphs with k edge-disjoint spanning trees, for any integer $k > 0$. For any graph G, the problem of determining which edges should be added to G so that the resulting graph has k edge-disjoint spanning trees has been studied; see [11,17], among others. However, it has not been fully studied that for an integer $k > 0$, if a graph G has k edge-disjoint spanning trees, what kind of edge $e \in E(G)$ has the property that $G - e$ also has k-edge-disjoint spanning trees. The research of this paper is motivated by this problem. In fact, we will consider the problem that, if a matroid M has k disjoint bases, what kind of element $e \in E(M)$ has the property that $M - e$ also has k disjoint bases.

We consider finite graphs with possible multiple edges and loops, and follow the notation of Bondy and Murty [1] for graphs, and Oxley [19] or Welsh [24] for matroids, except otherwise defined. Thus for a connected graph G, $\omega(G)$ denotes the number of components of G. For a matroid M, we use ρ_M (or ρ, when the matroid M is understood from the context) to denote the rank function of M, and $E(M)$, $\mathcal{C}(M)$ and $\mathcal{B}(M)$ to denote the ground set of M, and the collections of the circuits and the bases of M, respectively. Furthermore, if M is a matroid with $E = E(M)$, and if $X \subset E$, then $M - X$ is the restricted matroid of M obtained by deleting the elements in X from M, and M/X is the matroid obtained by contracting elements in X from M. As in [19,24], we use $M - e$ for $M - \{e\}$ and M/e for $M/\{e\}$.

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The *spanning tree packing number* of a connected graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. A survey on spanning tree packing number can be found in [20]. By definition, $\tau(K_1) = \infty$. For a matroid M, we similarly define $\tau(M)$ to be the maximum number of disjoint bases of M. Note that by definition, if M is a matroid with $\rho(M) = 0$, then for any integer $k > 0$, $\tau(M) \geq k$. The following theorems are well known.

Theorem 1.1 (*Nash-Williams [18] and Tutte [23]*). *Let G be a connected graph with $E(G) \neq \emptyset$, and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any $X \subseteq E(G)$, $|E(G - X)| \geq k(\omega(G - X) - 1)$.*

Theorem 1.2 (*Edmonds [9]*). *Let M be a matroid with $\rho(M) > 0$. Then $\tau(M) \geq k$ if and only if $\forall X \subseteq E(M)$, $|E(M) - X| \geq k(\rho(M) - \rho(X))$.*

Let M be a matroid with rank function r. For any subset $X \subseteq E(M)$ with $\rho(X) > 0$, the *density* of X is

$$d_M(X) = \frac{|X|}{\rho_M(X)}.$$

When the matroid M is understood from the context, we often omit the subscript M. We also use $d(M)$ for $d(E(M))$. Following the terminology in [4], the *strength* $\eta(M)$ and the *fractional arboricity* $\gamma(M)$ of M are respectively defined as

$$\eta(M) = \min\{d(M/X) : \rho(X) < \rho(M)\}, \quad \text{and} \quad \gamma(M) = \max\{d(X) : \rho(X) > 0\}.$$

Thus Theorem 1.2 above indicates that

$$\tau(M) = \lfloor \eta(M) \rfloor. \tag{1}$$

For an integer $k > 0$ and a matroid M with $\tau(M) \geq k$, we define $E_k(M) = \{e \in E(M) : \tau(M - e) \geq k\}$. Likewise, for a connected graph G with $\tau(G) \geq k$, $E_k(G) = \{e \in E(G) : \tau(G - e) \geq k\}$. Using Theorem 1.1, Gusfield proved that high edge-connectivity of a graph would imply high spanning tree packing number.

Theorem 1.3 (*Gusfield [10]*). *Let $k > 0$ be an integer, and let $\kappa'(G)$ denote the edge-connectivity of a graph G. If $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$.*

The next result strengthens Gusfield's theorem, and indicates a sufficient condition for a graph G to satisfy $E_k(G) = E(G)$.

Theorem 1.4 (*Theorem 1.1 of [5]*). *Let $k > 0$ be an integer, and let $\kappa'(G)$ denote the edge-connectivity of a graph G. Then $\kappa'(G) \geq 2k$ if and only if $\forall X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$. In particular, if $\kappa'(G) \geq 2k$, then $E_k(G) = E(G)$.*

A natural question is to characterize all graphs G with the property $E_k(G) = E(G)$. More generally, for any graph G with $\tau(G) \geq k$, we are to determine the edge subset $E_k(G)$. These questions can be presented in terms of matroids in a natural way. The main purpose of this paper is to characterize $E_k(M)$, for any matroid with $\tau(M) \geq k$. The next theorem is our main result.

Theorem 1.5. *Let M be a matroid and $k > 0$ be an integer. Each of the following holds.*

(i) *Suppose that $\tau(M) \geq k$. Then $E_k(M) = E(M)$ if and only if $\eta(M) > k$.*
(ii) *In general, $E_k(M)$ equals the maximal subset $X \subseteq E(M)$ such that $\eta(M|X) > k$.*

For a connected graph G with $M(G)$ denoting its cycle matroid, let $\eta(G) = \eta(M(G))$ and $\gamma(G) = \gamma(M(G))$. Then Theorem 1.5, when applied to cycle matroids, yields the corresponding theorem for graphs.

Corollary 1.6. *Let G be a connected graph and $k > 0$ be an integer. Each of the following holds.*

(i) *If $\tau(G) \geq k$, $E_k(G) = E(G)$ if and only if $\eta(G) > k$.*
(ii) *In general, $E_k(G)$ equals the maximal subset $X \subseteq E(G)$ such that every component of $\eta(G[X]) > k$.*

In the next section, we shall discuss properties of the strength and the fractional arboricity of a matroid M, which will be useful in the proofs of our main results. We will prove a decomposition theorem in Section 3, which will be applied in the characterizations of $E_k(M)$ and $E_k(G)$ in Section 4. In the last section, we shall develop polynomial algorithms to locate the sets $E_k(M)$ and $E_k(G)$.

2. Strength and fractional arboricity of a matroid

Both parameters $\eta(M)$ and $\gamma(M)$, and the problems related to uniformly dense graphs and matroids (defined below) have been studied by many; see [4,2,3,6,7,13–15,15,21,22], among others. From the definitions of $d(M)$, $\eta(M)$ and $\gamma(M)$, we immediately have, for any matroid M with $\rho(M) > 0$,

$$\eta(M) \leq d(M) \leq \gamma(M). \tag{2}$$