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Note

Maximal hypercubes in Fibonacci and Lucas cubes

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ABSTRACT

The Fibonacci cube Γ_n is the subgraph of the hypercube induced by the binary strings that contain no two consecutive 1's. The Lucas cube Λ_n is obtained from Γ_n by removing vertices that start and end with 1. We characterize maximal induced hypercubes in Γ_n and Λ_n and deduce for any $p \leq n$ the number of maximal p-dimensional hypercubes in these graphs.

1. Introduction

An interconnection topology can be represented by a graph $G = (V, E)$, where V denotes the processors and E the communication links. The *distance* $d_G(u, v)$ between two vertices u, v of a graph G is the length of a shortest path connecting u and v. An *isometric* subgraph H of a graph G is an induced subgraph such that for any vertices u, v of H we have $d_H(u, v) = d_G(u, v)$.

The *hypercube* of dimension n is the graph Q_n whose vertices are the binary strings of length n where two vertices are adjacent if they differ in exactly one coordinate. The *weight* of a vertex, $w(u)$, is the number of 1's in the string u. Notice that the graph distance between two vertices of Q_n is equal to the *Hamming distance* of the strings, the number of coordinates by which they differ. The hypercube is a popular interconnection network because of its structural properties.

Fibonacci cubes and Lucas cubes were introduced in [4,11] as new interconnection networks. They are isometric subgraphs of Q_n and have also recurrent structure.

A *Fibonacci string* of length n is a binary string $b_1b_2 \dots b_n$ with $b_ib_{i+1} = 0$ for $1 \leq i < n$. The *Fibonacci cube* Γ_n $(n \geq 1)$ is the subgraph of Q_n induced by the Fibonacci strings of length n. For convenience we also consider the empty string and set $\Gamma_0 = K_1$. Call a Fibonacci string $b_1b_2 \dots b_n$ a *Lucas string* if $b_1b_n \neq 1$. Then the *Lucas cube* Λ_n $(n \geq 1)$ is the subgraph of Q_n induced by the Lucas strings of length n. We also set $\Lambda_0 = K_1$.

Since their introduction, Γ_n and Λ_n have also been studied for their graph-theoretic properties [5,12] and found other applications, for example in chemistry (see the survey [7]). Recently different enumerative sequences of these graphs have been determined. Among them: the number of vertices of a given degree [10], the number of vertices of a given eccentricity [3], the number of pairs of vertices at a given distance [9] and the number of isometric subgraphs isomorphic to some Q_k [8]. The counting polynomial of this last sequence is known as *the cubic polynomial* and has very nice properties [1].

We propose to study another enumeration and characterization problem. For a given interconnection topology it is important to characterize maximal hypercubes, for example from the point of view of embeddings. So let us consider

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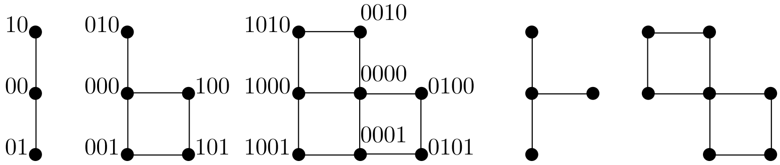

Fig. 1. $\Gamma_2 = \Lambda_2$, Γ_3, Γ_4 and Λ_3, Λ_4.

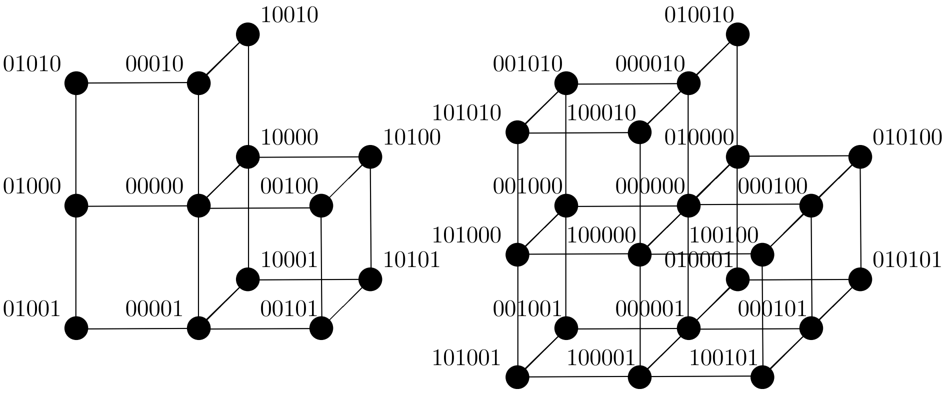

Fig. 2. Γ_5 and Γ_6.

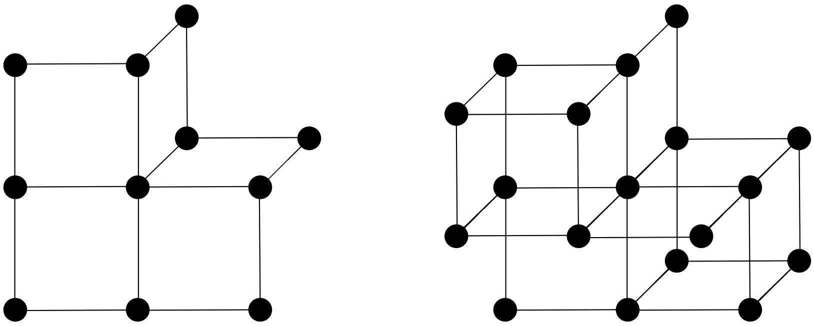

Fig. 3. Λ_5 and Λ_6.

maximal hypercubes of dimension p, i.e., induced subgraphs H of Γ_n (respectively Λ_n) that are isomorphic to Q_p, and such that there exists no induced subgraph H' of Γ_n (respectively Λ_n) isomorphic to Q_{p+1}, such that $H \subset H'$.

Let $f_{n,p}$ and $g_{n,p}$ be the numbers of maximal hypercubes of dimension p of Γ_n and Λ_n, respectively, and $C'(\Gamma_n, x) = \sum_{p=0}^{\infty} f_{n,p}x^p$ and $C'(\Lambda_n, x) = \sum_{p=0}^{\infty} g_{n,p}x^p$, respectively, their counting polynomials.

By direct inspection (see Figs. 1–3), we obtain the first of them:

$$C'(\Gamma_0, x) = 1 \qquad C'(\Lambda_0, x) = 1$$
$$C'(\Gamma_1, x) = x \qquad C'(\Lambda_1, x) = 1$$
$$C'(\Gamma_2, x) = 2x \qquad C'(\Lambda_2, x) = 2x$$
$$C'(\Gamma_3, x) = x^2 + x \qquad C'(\Lambda_3, x) = 3x$$
$$C'(\Gamma_4, x) = 3x^2 \qquad C'(\Lambda_4, x) = 2x^2$$
$$C'(\Gamma_5, x) = x^3 + 3x^2 \qquad C'(\Lambda_5, x) = 5x^2$$
$$C'(\Gamma_6, x) = 4x^3 + x^2 \qquad C'(\Lambda_6, x) = 2x^3 + 3x^2.$$

The intersection graph of maximal hypercubes (also called *the cube graph*) in a graph has been studied by various authors, for example in the context of median graphs [2]. Hypercubes play a role similar to cliques in clique graphs. Nice results have been obtained on the cube graph of median graphs, and it is thus of interest, from the graph-theoretic point of view, to characterize maximal hypercubes in families of graphs and thus obtain nontrivial examples of such graphs. We will first characterize maximal induced hypercubes in Γ_n and Λ_n and then deduce the number of maximal p-dimensional hypercubes in these graphs.