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On the sizes of graphs and their powers: The undirected case

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ABSTRACT

Let G be an undirected graph and G^r be its r-th power. We study different issues dealing with the number of edges in G and G^r. In particular, we answer the following question: given an integer $r \geq 2$ and all connected graphs G of order n such that $G^r \neq K_n$, what is the minimum number of edges that are added when going from G to G^r, and which are the graphs achieving this bound?

1. Introduction

Before we expound our study, we first give, for undirected graphs, some very basic definitions and notation, which can be found, e.g., in [4,7].

1.1. Definitions and notation

We shall denote by $G = (V, E)$ an *undirected graph* with vertex set V and edge set E, where an *edge* between $x \in V$ and $y \in V$ is indifferently denoted by the sets $\{x, y\}$ or $\{y, x\}$, x and y being called the *extremities* of the edge. We require the graph to have neither loops nor double edges. The *size* of a graph is its number of edges $|E|$, its *order* is its number of vertices $|V|$.

A *path* $P = x_0x_1 \ldots x_\ell$ is a sequence of vertices x_i, $0 \leq i \leq \ell$, such that $\{x_i, x_{i+1}\} \in E$, $0 \leq i \leq \ell - 1$. The *length* of P is its number of edges, ℓ. A graph is called *connected* if for any two vertices x and y there is a path between x and y.

In a connected graph G, we can define the distance between any vertex x and any vertex y, denoted by $d_G(x, y)$, as the number of edges in any shortest path between x and y, since such a path exists. Note that we have $d_G(x, y) = d_G(y, x)$. The *diameter* of a connected graph G is the maximum distance in the graph:

$$\mathrm{diam}(G) = \max_{x \in V, y \in V} d_G(x, y).$$

Definition. Given an integer $r \geq 1$, the *r-th power*, or *r-th transitive closure*, of the graph $G = (V, E)$ is the graph $G^r = (V, E^r)$, where, for two distinct vertices x and y, the edge $\{x, y\}$ belongs to E^r if and only if $d_G(x, y) \leq r$.

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The *clique*, or *complete graph*, K_n, is the graph of order n with all possible $n(n-1)/2$ edges. Finally, the *subgraph of* $G = (V, E)$ *induced by* $V^* \subseteq V$ is the graph $G^* = (V^*, E^*)$ where $E^* = \{\{x, y\} : x \in V^*, y \in V^*, \{x, y\} \in E\}$.

1.2. Scope of the paper

We are interested in the following interconnected problems on sizes and powers, for undirected graphs:

(**I**) Given the order and diameter of a connected graph, what is its maximum size, and which are the graphs achieving this bound?
The answer was given by Ore [14] as far back as 1968, see Section 2, and we will see that it actually ensues from the issue (III).

(**II**) Given an integer $r \geq 2$, what is the minimum size of a graph of order n, of which it is known that it is the r-th power of a connected graph, and which are the graphs achieving this bound?
The minimum size is already known [5, Sec. 9.3], see Theorem 2; what we do in Section 3.1 is to characterise the graphs which achieve the bound.
Almost unsolved is the following issue:

(**III**) Given an integer $r \geq 2$,
- (i) given all connected graphs G of order n such that $G^r \neq K_n$, what is the minimum number of edges that are added when going from G to G^r?
  The answer is known [1, Th. 1] only for $r = 2$, see Theorem 3.
- (ii) Which are the graphs achieving this bound?
- (iii) How many edges can we have in the graphs reaching the bound?
  This is the unanswered 10-year-old Question 1 in [1], formulated only for $r = 2$.

We give the complete answers to question (III) in Section 3.2, which is the core part of our article.

Similar issues for directed graphs are treated in [2].

The relationships between graph parameters, such as order, diameter and size as in question (I), have been widely studied, both in undirected and directed graphs; see, e.g., [3, Sec. 2.4], [8,10–12,14].

The study of graphs and their powers can be best illustrated in problems where the distance in the graph is crucial. For instance, looking for a set of vertices whose pairwise distances are at least 3 in G amounts to finding a stable set in G^2, and an r-identifying code in G is a 1-identifying code in G^r (see for instance [6] or more generally [13] on identifying codes). In addition to this, squares of graphs may have Hamiltonian properties: Fleischner [9] proved in 1974 that the square of a 2-vertex-connected graph admits a Hamilton cycle.

Another possible illustration for $r = 2$ is the following. The vertices represent persons, and edges friendship between two persons; assume that you gather everyone and that two persons with a common friend become friends: how many friendships can be created?

2. State of the art

(**I**) As aforementioned, the maximum size of a graph of given order and diameter is known, as well as the families of graphs which achieve this bound.

Theorem 1 (*[14], Th. 3.1*). *Let $G = (V, E)$ be a connected graph of order n and diameter $\delta \geq 2$. Then the size of G is at most*

$$\delta + \frac{(n-\delta-1)(n-\delta+4)}{2}, \tag{1}$$

and this bound is tight for some graphs.

Proof. We give a very simple proof, different from that by Ore. Let $z_1, z_2 \in V$ be such that $d_G(z_1, z_2) = \delta$, and P be the shortest path between them: $P = x_0x_1 \dots x_\delta$, with $x_0 = z_1$ and $x_\delta = z_2$. Notice that there are no more edges between vertices x_i, otherwise P would not be the shortest path. In G, the remaining vertices y_j, $1 \leq j \leq n-\delta-1$, can at most constitute the clique $K_{n-\delta-1}$, and each y_j can be part of at most three edges with extremities in P. This is obvious if $\delta = 2$, and if $\delta \geq 3$, there would be otherwise two edges $\{y_j, x_{i_1}\}$ and $\{y_j, x_{i_2}\}$ with $i_1 + 3 \leq i_2$, and the path $x_0 \dots x_{i_1}y_jx_{i_2} \dots x_\delta$ would be shorter than P. Summing up, we have at most

$$\delta + \frac{1}{2}(n-\delta-1)(n-\delta-2) + 3(n-\delta-1)$$

edges in G, which amounts to (1). □

We shall see that this theorem is also a direct consequence of Theorem 7 introduced later in the paper, in Section 3.2.1. We set

$$s(\delta, n) = \delta + \frac{(n-\delta-1)(n-\delta+4)}{2};$$