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Paths, trees and matchings under disjunctive constraints[☆,☆☆]

Andreas Darmann[a], Ulrich Pferschy[b,*], Joachim Schauer[b], Gerhard J. Woeginger[c]

[a] *University of Graz, Institute of Public Economics, Universitaetsstr. 15, A-8010 Graz, Austria*

[b] *University of Graz, Department of Statistics and Operations Research, Universitaetsstr. 15, A-8010 Graz, Austria*

[c] *TU Eindhoven, Department of Mathematics and Computer Science, P.O. Box 513, NL-5600 MB Eindhoven, The Netherlands*

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ABSTRACT

We study the minimum spanning tree problem, the maximum matching problem and the shortest path problem subject to binary disjunctive constraints: A *negative disjunctive constraint* states that a certain pair of edges cannot be contained simultaneously in a feasible solution. It is convenient to represent these negative disjunctive constraints in terms of a so-called *conflict graph* whose vertices correspond to the edges of the underlying graph, and whose edges encode the constraints.

We prove that the minimum spanning tree problem is strongly $\mathcal{NP}$-hard, even if every connected component of the conflict graph is a path of length two. On the positive side, this problem is polynomially solvable if every connected component is a single edge (that is, a path of length one). The maximum matching problem is $\mathcal{NP}$-hard for conflict graphs where every connected component is a single edge.

Furthermore we will also investigate these graph problems under *positive disjunctive constraints*: In this setting for certain pairs of edges, a feasible solution must contain at least one edge from every pair. We establish a number of complexity results for these variants including APX-hardness for the shortest path problem.

1. Introduction

We study variants of the minimum spanning tree problem (MST), of the maximum matching problem (MM) and of the shortest path problem (SP) in weighted, undirected, connected graphs. These variants are built around binary disjunctive constraints on certain pairs of edges.

- A negative disjunctive constraint expresses an incompatibility or a conflict between the two edges in a pair. From each conflicting pair, at most one edge can occur in a feasible solution.
- A positive disjunctive constraint enforces that at least one edge from the underlying pair is in a feasible solution.

Throughout we will represent these binary disjunctive constraints by means of an undirected constraint graph: Every vertex of the constraint graph corresponds to an edge in the original graph, and every edge corresponds to a binary constraint. In the case of negative disjunctive constraints this constraint graph is usually called a *conflict graph* (e.g [10,8,7,4]) and in the case of positive disjunctive constraints this graph will be called a *forcing graph*.

☆ Parts of this paper extend earlier work on the minimum spanning tree problem with conflict graphs which appeared as Darmann et al. (2009) [5].

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* Corresponding author.
E-mail addresses: andreas.darmann@uni-graz.at (A. Darmann), pferschy@uni-graz.at (U. Pferschy), joachim.schauer@uni-graz.at (J. Schauer), gwoegi@win.tue.nl (G.J. Woeginger).

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For a formal definition of the *minimum spanning tree problem with conflict graph* (MSTCG), the *maximum matching problem with conflict graph* (MMCG) and the *shortest path problem with conflict graph* (SPCG), let $G = (V, E)$ be an undirected connected graph with n vertices and m edges. Every edge e has a weight $w(e)$ (where w is a weight function $w : E \to \mathbb{R}$). Furthermore, the undirected graph $\bar{G} = (E, \bar{E})$ represents the conflict graph where each of the m vertices corresponds uniquely to an edge $e \in E$ of G. An edge $\bar{e} = (i, j) \in \bar{E}$ implies that the two vertices incident to $\bar{e}$ – that is, the two edges $i, j \in E$ – cannot occur together in a spanning tree or maximum matching of G. In contrast to graph G, the conflict graph $\bar{G}$ is not necessarily connected and may contain isolated vertices (i.e. edges of G which can be combined with all other edges in the minimum spanning tree solution). MSTCG asks for a minimum spanning tree T in G, given that adjacent vertices in $\bar{G}$ are not both together included in T, MMCG asks for a maximum matching in G, given that adjacent vertices in $\bar{G}$ do not both belong to the maximum matching and SPCG asks for a shortest *simple* path in G, given that adjacent vertices in $\bar{G}$ do not both belong to the shortest path.

Similarly, we define the problems *minimum spanning tree problem with forcing graph* (MSTFG), *maximum matching problem with forcing graph* (MMFG) and *shortest path problem with forcing graph* (SPFG) around positive disjunctive constraints. Every vertex of a forcing graph $\bar{H} = (E, \bar{E})$ corresponds to an edge $e \in E$ of G, and an edge $\bar{e} = (i, j) \in \bar{E}$ implies that at least one of the two vertices incident to $\bar{e}$ – that is, at least one of the two edges $i, j \in E$ – has to be included in a spanning tree or maximum matching of G. Again the graph $\bar{H}$ is not necessarily connected and may contain isolated vertices.

Note that for all considered problems MSTCG, MMCG, MSTFG, MMFG, SPCG and SPFG the existence of a feasible solution is not at all guaranteed.

In this paper we will characterize the complexity of MSTCG, MMCG, MSTFG, MMFG, and SPFG and we will identify graph classes for the conflict (forcing) graph $\bar{G}$ ($\bar{H}$) where the computationally complexity jumps from polynomially solvable to strongly $\mathcal{NP}$-hard. For illustrative reasons we introduce the following terminology.

Definition 1. A 2-ladder is an undirected graph whose components are paths of length one, i.e. edges connecting pairs of vertices.

Definition 2. A 3-ladder is an undirected graph whose components are paths of length two.

Results of this paper

For the minimum spanning tree problem we establish a sharp separation line between easy and hard instances. The results of Sections 2 and 4 establish that problems MSTCG and MSTFG are strongly $\mathcal{NP}$-hard, even if the underlying conflict (forcing) graph is a 3-ladder. On the other hand, we show by a matroid intersection argument in Section 3 and in Section 5 that the minimum spanning tree problem is polynomially solvable for a 2-ladder as a conflict (forcing) graph.

The considered variants of the maximum matching problem seem to be universally hard: Sections 6 and 7 show that problems MMCG and MMFG are strongly $\mathcal{NP}$-hard even for 2-ladder conflict (forcing) graphs.

The shortest path problem with conflict graphs is known to be NPO PB-complete [9], even if the conflict graph is a 2-ladder. For SPFG we will show in Section 8 as a complementary result that this problem is already APX-hard for a 2-ladder as a forcing graph. Note that the results for SPCG and SPFG hold even for the unweighted case where the number of edges of the path is minimized.

Related results

Results of a similar flavor have been derived recently for the 0–1 knapsack problem with conflict graphs. While this problem is strongly $\mathcal{NP}$-hard for arbitrary conflict graphs, it was shown in [10] that pseudopolynomial algorithms (and hence also fully polynomial approximation schemes) exist if the given conflict graph is a tree, a graph of bounded treewidth or a chordal graph. Bin packing problems with special classes of conflict graphs were considered from an approximation point of view by [8,7]. Complexity results for different classes of conflict graphs for a scheduling problem under makespan minimization are given in [4]. Recently the problem was considered for the maximum flow problem [11]. Further references on combinatorial optimization problems with conflict graphs can be found in [10].

2. A strong $\mathcal{NP}$-hardness result for MSTCG

In this section we show that MSTCG is strongly $\mathcal{NP}$-hard even if the conflict graph $\bar{G}$ is a 3-ladder. As an example, consider a component of $\bar{G}$ that consists of the path $(e_1e_2e_3)$ on the three edges $e_1, e_2, e_3 \in E$: If a feasible spanning tree for the underlying graph G contains the edge e_2, then it must neither include edge e_1 nor edge e_3. And if a feasible tree contains edge e_1, then it must not contain e_2, but may contain e_3.

2.1. The graphs G_{MSTCG} and $\bar{G}_{\text{MSTCG}}$

We reduce the NP-complete problem (3, *B*2)-*SAT* [3] to special instances of MSTCG which are described by a graph G_{MSTCG} in which a spanning tree has to be found subject to a conflict graph $\bar{G}_{\text{MSTCG}}$. (3, *B*2)-*SAT* is the special symmetric subproblem of 3-*SAT* in which each clause has size three and each literal occurs exactly twice. This means that each variable occurs exactly four times, twice negated and twice nonnegated.