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Note

Girth of pancake graphs

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ABSTRACT

We consider four families of pancake graphs, which are Cayley graphs, whose vertex sets are either the symmetric group on n objects or the hyperoctahedral group on n objects and whose generating sets are either all reversals or all reversals inverting the first k elements (called prefix reversals). We find that the girth of each family of pancake graphs remains constant after some small threshold value of n.

1. Introduction

A *signed permutation on n objects* is a function

$$\alpha\colon \{1, 2, \ldots, n\} \to \{-n, \ldots, -1\} \cup \{1, \ldots, n\}$$

such that $|\alpha|$ is in S_n. We represent a signed permutation α as an n-tuple

$$\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$$

and we can think about α as a permutation on n objects in which each object is provided a sign. For two signed permutations on n objects, say α and β, we define the composition of β with α by

$$(\beta\alpha)(i) := \alpha(|\beta(i)|) \cdot \operatorname{sgn}\beta(i).$$

For example, $(-4, 1, -3, -2)(1, 3, -4, -2) = (2, 1, 4, -3)$. Under this operation, the set of all signed permutations on n objects forms the *hyperoctahedral group on n objects*, which we denote by B_n. This group is commonly known as the group of symmetries of the n-dimensional hypercube.

In this light, we shall call the members of S_n *unsigned permutations on n objects*. As with composition of signed permutations, our composition in S_n will be written left-to-right:

$$(\beta\alpha)(i) := \alpha(\beta(i)).$$

Henceforth, "permutation" will refer generally to both signed and unsigned permutations. The identity of both S_n and B_n is denoted by I_n.

For fixed n and $1 \le j < k \le n$, the *unsigned reversal on the interval* $[j, k]$ is the permutation $\upsilon_{[j,k]} \in S_n$ defined by

$$\upsilon_{[j,k]} := (1, 2, \ldots, j-1, k, k-1, \ldots, j, k+1, k+2, \ldots, n).$$

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In the case that $j = 1$ (and thus $2 \le k \le n$), we write $\upsilon_{[k]} := \upsilon_{[1,k]}$ and say that $\upsilon_{[k]}$ is the *unsigned prefix reversal at index k*. Let Υ_n (Υ_n^p) denote the set of all unsigned reversals (unsigned prefix reversals) in S_n.

Analogously, for $1 \le j \le k \le n$, the *signed reversal on the interval* $[j, k]$ is the permutation $\sigma_{[j,k]} \in B_n$ defined by

$$\sigma_{[j,k]} := (1, 2, \dots, j-1, -k, -(k-1), \dots, -j, k+1, k+2, \dots, n).$$

Note that in the signed case we allow $k = j$, since for all $1 \le j \le n$, $\sigma_{[j,j]} \ne I_n$. Furthermore, for $1 \le k \le n$, the *signed prefix reversal at index k* is the permutation $\sigma_{[k]} := \sigma_{[1,k]}$. Let Σ_n (Σ_n^p) denote the set of all signed reversals (signed prefix reversals) in B_n. Henceforth, "reversal" may refer to any signed or unsigned reversal, prefix or otherwise, and we use ρ to denote an arbitrary reversal.

Next, define the *unsigned reversal graph* (*unsigned prefix reversal graph*) on n objects, denoted by UR_n (UP_n), as the Cayley graph whose vertex set is S_n and whose generating set is Υ_n (Υ_n^p). Both graphs have order $|S_n| = n!$; UP_n has degree $n-1$, while UR_n has degree $\binom{n}{2}$.

Analogously, define the *signed reversal graph* (*signed prefix reversal graph*) on n objects, denoted by SR_n (SP_n), as the Cayley graph with vertex set B_n and generating set Σ_n (Σ_n^p). Both graphs have order $|B_n| = 2^n \cdot n!$; SP_n has degree n, while SR_n has degree $\binom{n+1}{2}$.

We refer to all four families of Cayley graphs collectively as *pancake graphs*. Before continuing, we will give two straightforward facts about pancake graphs. We will see later that the second fact will provide an upper bound on the girth of each family of pancake graphs once we have found a short cycle in that family.

Fact 1. *For any n, $\Upsilon_n^p \subset \Upsilon_n$ and $\Sigma_n^p \subset \Sigma_n$, so that UP_n is a subgraph of UR_n and SP_n is a subgraph of SR_n.*

Fact 2. *Every pancake graph embeds in all higher-order pancake graphs of the same family. For example, if $m \le n$, then UP_m is isomorphic to the Cayley subgraph of UP_n generated by the subset of Υ_n^p containing only those unsigned prefix reversals υ_k for which $k \le m$.*

The etymology of "pancake graph" traces back to a 1975 *American Mathematical Monthly* problem, which asked for a function $f(n)$ bounding the maximum number of flips required to transform a given stack of n differently sized pancakes into the stack whose pancakes are sorted from top to bottom in the order of increasing size. Of course, stacks of pancakes correspond to unsigned permutations; if all the pancakes are burned on one side (creating the "burned pancake problem"), stacks correspond to signed permutations. A pancake graph is therefore a graph whose vertices are stacks of n pancakes, and whose edges represent flips between stacks: prefix reversals constitute a "one-spatula" case and reversals constitute a "two-spatula" case. For three decades, the best known bound for the pancake problem was found in [6], although it has recently been improved in [3]. See [1,2,4,5,7,9] and [10] for more on the pancake problem and its offshoots.

A biological application of pancake flipping is found in genetic analysis. One common form of large-scale evolutionary change is a genomic mutation which manifests itself in the reversal of some segment of the mutated organism's DNA. Phylogeneticists study how the accumulation of millions of years of mutations, including reversals of DNA segments, have led to species divergence. Therefore, a given property of pancake graphs can in many cases be translated into a phylogenetic application when only reversal mutations are considered. For example, it seems extremely unlikely that evolutionary changes occur in cycles; however, if pancake graphs were to be shown to have large girth (say $O(n)$), then we would have concrete evidence that cyclical evolutionary patterns are implausible.

2. Preliminaries

In this note, our aim is to find the girth of each of the four families of pancake graphs introduced above. For $k \ge 3$, we define a *cycle of length k* in a pancake graph as a reduced finite sequence of reversals $(\rho_1, \rho_2, \dots, \rho_k)$, all of which lie in the appropriate generating set, and such that $\rho_k \cdots \rho_1 = I_n$. By a "reduced sequence", we mean a sequence for which $\rho_1 \ne \rho_k$ and $\rho_{i+1} \ne \rho_i$ for all $1 \le i \le k-1$, since all reversals are involutions within their respective permutation groups. Observe that this definition of cycle agrees with the graph theoretical one; therefore, the girth of a pancake graph will be the minimal length of a cycle of reversals taken from that graph's generating set.

Let $\alpha \in S_n$ and extend α to a member of S_{n+2} by setting $\alpha(0) = 0$ and $\alpha(n+1) = n+1$. For $0 \le i \le n$, we continue the terminology established in [9] and say that α has a *breakpoint at i* if $|\alpha(i+1) - \alpha(i)| \ne 1$. In the signed case, we also extend $\beta \in B_n$ to an element of B_{n+2} by setting $\beta(0) = 0$ and $\beta(n+1) = n+1$. In this case, however, we say that β has a breakpoint at i if $\beta(i+1) - \beta(i) \ne 1$. For example, the signed permutation $\beta = (-4, -3, -2, 1, 5)$ has breakpoints at 0, 3, and 4.

We also define a *non-initial breakpoint* of a permutation to be any breakpoint other than the breakpoint at 0. Observe that in both the signed and unsigned cases, the identity I_n is the only permutation with no breakpoints. In pancake flipping, it is therefore interesting to think of breakpoints as something we wish to eliminate during a walk to the identity.

A set $B \subset B_n$ will be called k-compressible if for some $J \subset \{0, 1, 2, \dots, n\}$ with $|J| = k$, B contains only signed permutations β such that for every $j \in J$, either $\beta^{-1}(j+1) - \beta^{-1}(j) = 1$ or $\beta^{-1}(-j) - \beta^{-1}(-(j+1)) = 1$. (In other words, if j occurs in β, then $j+1$ occurs immediately to the right of j in β, and otherwise $-(j+1)$ occurs immediately to the