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Almost Hadamard matrices: The case of arbitrary exponents

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ABSTRACT

A square matrix $H \in M_N(\mathbb{R})$ is called "almost Hadamard" if $U = H/\sqrt{N}$ is orthogonal, and locally maximizes the 1-norm on $O(N)$. We review our previous work on the subject, notably with the formulation of a new question, regarding the circulant and symmetric case. We discuss then an extension of the almost Hadamard matrix formalism, by making use of the p-norm on $O(N)$, with $p \in [1, \infty] - \{2\}$, with a number of theoretical results on the subject, and the formulation of some open problems.

0. Introduction

An Hadamard matrix is a square matrix $H \in M_N(\pm 1)$ having its rows pairwise orthogonal. The Hadamard conjecture (HC), which is over a century old, states that such matrices exist, at any $N \in 4\mathbb{N}$. See [1,18,22,14]. The circulant Hadamard conjecture (CHC), which is half a century old [25], states that a circulant Hadamard matrix can exist only at $N = 4$. More precisely, only the following matrix K_4 and its various "conjugates" can be at the same time circulant and Hadamard, regardless of the size $N \in \mathbb{N}$:

$$K_4 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

An interesting generalization of the Hadamard matrices are the complex Hadamard matrices, namely the matrices $H \in M_N(\mathbb{T})$, where $\mathbb{T}$ is the unit circle, having their rows pairwise orthogonal. These matrices appear in several contexts, see [16,19,20,24,28,30,31]. The main example is the rescaled Fourier matrix ($w = e^{2\pi i/N}$):

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{pmatrix}.$$

This example prevents the existence of a complex analogue of the HC. However, when trying to build complex Hadamard matrices with roots of unity of a given order, a subtle generalization of the HC problematics appears [10,23,13]. In relation

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now with the CHC, there has been some interesting work here on the circulant case [8,9,17]. Also, much work has gone into various geometric aspects, see [2,6,21,27,29].

Yet another generalization comes from [3,4]. The original observation from [3] is that for an orthogonal matrix $U \in O(N)$ we have $\|U\|_1 \leq N\sqrt{N}$, with equality if and only if $H = \sqrt{N}U$ is Hadamard. This follows indeed from the Cauchy–Schwarz inequality:

$$\|U\|_1 = \sum_{ij} |U_{ij}| \leq N \left(\sum_{ij} U_{ij}^2 \right)^{1/2} = N\sqrt{N}.$$

This simple fact suggests that a natural and useful generalization of the Hadamard matrices are the matrices of type $H = \sqrt{N}U$, with $U \in O(N)$ being a maximizer of the 1-norm. However, since such matrices are quite difficult to approach, most efficient is to study first the matrices of type $H = \sqrt{N}U$, with $U \in O(N)$ being just a local maximizer of the 1-norm. Such matrices are called "almost Hadamard". See [4].

One key feature of the almost Hadamard matrices is that at the level of examples we have a number of infinite series, uniformly depending on $N \in \mathbb{N}$. The basic example is:

$$K_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 2-N & 2 & \cdots & 2 \\ 2 & 2-N & \cdots & 2 \\ \cdots & & & \\ 2 & 2 & \cdots & 2-N \end{pmatrix}.$$

Observe that K_N is circulant, and that K_4 is Hadamard. Thus we are quickly led into the circulant Hadamard matrix problematics, and we have the following questions:

Problem. What are the circulant Hadamard matrices? The circulant complex Hadamard matrices? The circulant almost Hadamard matrices?

More precisely, the CHC states that there are exactly 8 circulant Hadamard matrices, namely K_4 and its conjugates. Regarding the second question, Haagerup has shown in [17] that for $N = p$ prime, the number of circulant complex Hadamard matrices, counted with certain multiplicities, is exactly $\binom{2p-2}{p-1}$, and the problem is to see what happens when N is not prime. As for the third question, this appears from our previous work [4].

Regarding this latter question, it was shown in [4] that we have a number of interesting examples coming from block designs [11,26]. The simplest one, coming from the adjacency matrix of the Fano plane, is as follows, with $x = 2-4\sqrt{2}$, $y = 2+3\sqrt{2}$: (see Fig. 1)

$$I_7 = \frac{1}{2\sqrt{7}} \begin{pmatrix} x & x & y & y & y & x & y \\ y & x & x & y & y & y & x \\ x & y & x & x & y & y & y \\ y & x & y & x & x & y & y \\ y & y & x & y & x & x & y \\ y & y & y & x & y & x & x \\ x & y & y & y & x & y & x \end{pmatrix}.$$

Now back to the above 3 questions, the point is that, from the point of view of Fourier analysis, these are all related. Indeed, with $F = F_N/\sqrt{N}$, the circulant unitary matrices are precisely those of the form $U = FQF^*$ with Q belonging to the torus $\mathbb{T}^N$ formed by the diagonal matrices over $\mathbb{T}$. So, in view of the above-mentioned remark about the 1-norm, all the above questions concern the understanding of the following potential:

$$\begin{aligned} &\Phi : \mathbb{T}^N \to [0, \infty) \\ &Q \to \|FQF^*\|_1. \end{aligned}$$

With this approach, the first thought goes to the computation of the moments of Φ. Indeed, the global maximum, or more specialized quantities such as the exact number of maxima, can be recovered via variations of the following well-known formula:

$$\max(\Phi) = \lim_{k\to\infty} \left(\int_{\mathbb{T}^N} \Phi^k \right)^{1/k}.$$

Of course, in respect to the above problems, one has to restrict sometimes attention to the torus $\mathbb{T}^n \subset \mathbb{T}^N$, with $n = [(N+1)/2]$, coming from the orthogonal matrices.

The origins of this approach go back to [3], where the potential $\Phi(U) = \|U\|_1$ was investigated over the group $O(N)$, in connection with the HC. Of course, the computation of moments over $O(N)$ is a quite complicated question [5,12]. In the circulant case, however, the parameter space being just $\mathbb{T}^N$, the integration problem is much simpler. But it still remains very complicated, and we have no concrete results here so far.