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Finding a smallest odd hole in a claw-free graph using global structure

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ABSTRACT

A lemma of Fouquet implies that a claw-free graph contains an induced C_5, contains no odd hole, or is quasi-line. In this paper, we use this result to give an improved shortest-odd-hole algorithm for claw-free graphs by exploiting the structural relationship between line graphs and quasi-line graphs suggested by Chudnovsky and Seymour's structure theorem for quasi-line graphs. Our approach involves reducing the problem to that of finding a shortest odd cycle of length ≥ 5 in a graph. Our algorithm runs in $O(m^2 + n^2 \log n)$ time, improving upon Shrem, Stern, and Golumbic's recent $O(nm^2)$ algorithm, which uses a local approach. The best known recognition algorithms for claw-free graphs run in $O(m^{1.69}) \cap O(n^{3.5})$ time, or $O(m^2) \cap O(n^{3.5})$ without fast matrix multiplication.

1. Background and motivation

A *hole* in a graph is an induced cycle C_k of length $k \geq 4$. Odd holes are fundamental to the study of perfect graphs [5]; although there are polynomial-time algorithms that decide whether or not either a graph or its complement contains an odd hole [9,2], no general algorithm for detecting an odd hole in a graph is known.

Odd holes are also fundamental to the study of claw-free graphs, i.e. graphs containing no induced copy of $K_{1,3}$. Every neighbourhood v in a claw-free graph has stability number $\alpha(G[N(v)]) \leq 2$. So if $G[N(v)]$ is perfect then v is bisimplicial (i.e. its neighbours can be partitioned into two cliques, i.e. $G[N(v)]$ is cobipartite), and if $G[N(v)]$ is imperfect then $G[N(v)]$ contains the complement of an odd hole. Fouquet proved something stronger.

Lemma 1 (*Fouquet [11]*). *Let G be a connected claw-free graph with $\alpha(G) \geq 3$. Then every vertex of G is bisimplicial or contains an induced C_5 in its neighbourhood.*

It follows that a claw-free graph G has $\alpha(G) \leq 2$, or contains an induced C_5 in the neighbourhood of some vertex, or is *quasi-line*, meaning every vertex is bisimplicial.

Chvátal and Sbihi proved a decomposition theorem for perfect claw-free graphs that yields a polynomial-time recognition algorithm [8]. More recently, Shrem, Stern, and Golumbic gave an $O(nm^2)$ algorithm for finding a shortest odd hole in a claw-free graph based on a variant of breadth-first search in an auxiliary graph [18]. We solve the same problem, but instead of using local structure we use global structure and take advantage of the similarities between claw-free graphs, quasi-line graphs, and line graphs. We prove the following theorem.

Theorem 2. *There exists an algorithm that, given a claw-free graph G on n vertices and m edges, finds a smallest odd hole in G or determines that none exists in $O(m^2 + n^2 \log n)$ time.*

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Fouquet's lemma allows us to focus on quasi-line graphs. Their global structure, described by Chudnovsky and Seymour [6], resembles that of line graphs closely enough that we can reduce the shortest odd hole problem on quasi-line graphs to a set of shortest path problems in underlying multigraphs. Our algorithm is not much slower than the fastest known recognition algorithms for claw-free graphs: Alon and Boppana gave an $O(n^{3.5})$ recognition algorithm [1]. Kloks, Kratsch, and Müller gave an $O(m^{1.69})$ recognition algorithm that relies on impractical fast matrix multiplication [15]. Their approach takes $O(m^2)$ time using naïve matrix multiplication, and more generally $O(m^{(\beta+1)/2})$ time using $O(n^\beta)$ matrix multiplication.

2. The easy cases: finding a C_5

We begin by taking advantage of Fouquet's lemma in order to reduce the problem to quasi-line graphs. We denote the closed neighbourhood of a vertex v by $\tilde{N}(v)$.

Theorem 3. *Let G be graph with $\alpha(G) \leq 2$. In $O(m^2)$ time we can find an induced C_5 in G or determine that none exists.*

Proof. For each edge uv we do the following. First, we construct sets $X = N(u) \setminus \tilde{N}(v)$, $Y = N(v) \setminus \tilde{N}(u)$, and $Z = V(G) \setminus (N(u) \cup N(v))$. If u and v are in an induced C_5 together then all three must be nonempty. Since $\alpha(G) \leq 2$, we know that both X and Y are complete to Z. Second, we search for $x \in X$ and $y \in Y$ which are nonadjacent – if such x and y exist then this clearly gives us C_5. It is easy to see that we can construct the sets in $O(n)$ time, and that we can search for a non-edge between X and Y in $O(m)$ time, since we can terminate once we find one. Thus it takes $O(m^2)$ time to do this for every edge, and if an induced C_5 exists in G we will identify it as $uvyzx$ for any $z \in Z$. □

Kloks, Kratsch, and Müller observed that as a consequence of Turán's theorem, every vertex in a claw-free graph has at most $2\sqrt{m}$ neighbours [15]. We make repeated use of this fact, starting with a consequence of the previous lemma.

Corollary 4. *Let G be a claw-free graph with $\alpha(G) \geq 3$. Then in $O(m^2)$ time we can find an induced W_5 in G or determine that G is quasi-line.*

Proof. By Fouquet's lemma, any vertex of G is either bisimplicial or contains an induced C_5 in its neighbourhood. For any $v \in V(G)$, we can easily check whether or not $G[N(v)]$ is cobipartite in $O(d(v)^2)$ time. Since G is claw-free, $d(v)^2 = O(m)$. Thus in $O(nm)$ time we can determine that G is quasi-line or find a vertex v which is not bisimplicial.

Given this v, we can find an induced C_5 in $G[N(v)]$ in $O(m^2)$ time by applying the method in the previous proof, since $\alpha(G[N(v)]) \leq 2$. □

Having dealt with these cases made easy by Fouquet's lemma, we can move on to quasi-line graphs with $\alpha \geq 3$, the structure of which we describe now.

3. The structure of quasi-line graphs

Given a multigraph H (with loops permitted), its *line graph* $L(H)$ is the graph with one vertex for each edge of H, in which two vertices are adjacent precisely if their corresponding edges in H share at least one endpoint. Thus the neighbours of any vertex v in $L(H)$ are covered by two cliques, one for each endpoint of the edge in H corresponding to v. We say that G is a line graph if $G = L(H)$ for some multigraph H.

Chudnovsky and Seymour [6] described exactly how quasi-line graphs generalize line graphs: a quasi-line graph is essentially either a circular interval graph or can be obtained from a multigraph by replacing each edge with a linear interval graph.

3.1. Linear and circular interval graphs

A *linear interval graph* is a graph $G = (V, E)$ with a *linear interval representation*, which is a point on the real line for each vertex and a set of intervals, such that vertices u and v are adjacent in G precisely if there is an interval containing both corresponding points on the real line. If X and Y are specified cliques in G consisting of the $|X|$ leftmost and $|Y|$ rightmost vertices (with respect to the real line) of G respectively, we say that X and Y are *end-cliques* of G. Given a linear interval representation, if u is to the left of v we say that $u < v$. If u and v are adjacent, we say that u is a *left neighbour* of v, and v is a *right neighbour* of u.

Accordingly, a *circular interval graph* is a graph with a *circular interval representation*, i.e. $|V|$ points on the unit circle and a set of intervals (arcs) on the unit circle such that two vertices of G are adjacent precisely if some arcs contain both corresponding points. For distinct u and v in V, if there is an arc containing u, v, and all points on the circle reached by moving clockwise (resp. counterclockwise) from u until reaching v, we say that v is a *clockwise neighbour* (resp. *counterclockwise neighbour*) of u. Circular interval graphs are the first of two fundamental types of quasi-line graph. Deng, Hell, and Huang proved that we can identify and find a representation of a circular or linear interval graph in $O(m)$ time [10].