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Bicyclic graphs with maximal revised Szeged index[☆]

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ABSTRACT

The revised Szeged index of a graph G is defined as $Sz^*(G) = \sum_{e=uv\in E}(n_u(e) + n_0(e)/2)(n_v(e)+n_0(e)/2)$, where $n_u(e)$ and $n_v(e)$ are, respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u, and $n_0(e)$ is the number of vertices equidistant to u and v. Hansen et al. used the AutoGraphiX and made the following conjecture about the revised Szeged index for a connected bicyclic graph G of order $n \geq 6$:

$$Sz^*(G) \leq \begin{cases} (n^3 + n^2 - n - 1)/4, & \text{if } n \text{ is odd,} \\ (n^3 + n^2 - n)/4, & \text{if } n \text{ is even.} \end{cases}$$

with equality if and only if G is the graph obtained from the cycle C_{n-1} by duplicating a single vertex. This paper is to give a confirmative proof to this conjecture.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. We refer the readers to [2] for terminology and notations. Let G be a connected graph with vertex set V and edge set E. For $u, v \in V$, $d(u, v)$ denotes the distance between u and v. The *Wiener index* of G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V} d(u, v).$$

This topological index has been extensively studied in the mathematical literature; see, e.g., [4,6]. Let $e = uv$ be an edge of G, and define three sets as follows:

$$N_u(e) = \{w \in V : d(u, w) < d(v, w)\},$$
$$N_v(e) = \{w \in V : d(v, w) < d(u, w)\},$$
$$N_0(e) = \{w \in V : d(u, w) = d(v, w)\}.$$

Thus, $\{N_u(e), N_v(e), N_0(e)\}$ is a partition of the vertices of G with respect to e. The number of vertices of $N_u(e)$, $N_v(e)$ and $N_0(e)$ are denoted by $n_u(e)$, $n_v(e)$ and $n_0(e)$, respectively. A long time known property of the Wiener index is the formula [5,12]:

$$W(G) = \sum_{e=uv\in E} n_u(e)n_v(e),$$

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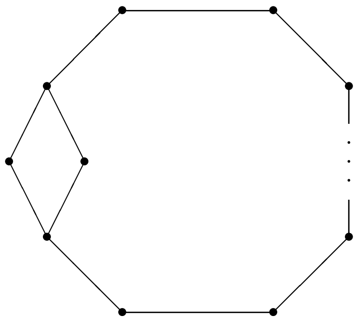

Fig. 1. B_n.

which is applicable for trees. Using the above formula, Gutman [3] introduced a graph invariant named the *Szeged index* as an extension of the Wiener index and defined it by

$$Sz(G) = \sum_{e=uv\in E} n_u(e)n_v(e).$$

Randić [10] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named the *revised Szeged index*. The revised Szeged index of a connected graph G is defined as

$$Sz^*(G) = \sum_{e=uv\in E} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right).$$

Some properties and applications of these two topological indices have been reported in [8,9,11]. In [1], Aouchiche and Hansen showed that for a connected graph G of order n and size m, an upper bound of the revised Szeged index of G is $\frac{n^2m}{4}$. In [13], Xing and Zhou determined the unicyclic graphs of order n with the smallest and the largest revised Szeged indices for $n \geq 5$, and they also determined the unicyclic graphs of order n with a unique cycle of length r ($3 \leq r \leq n$), with the smallest and the largest revised Szeged indices.

In [7], Hansen et al. used the AutoGraphiX and made the following conjecture.

Conjecture 1.1. *Let G be a connected bicyclic graph G of order $n \geq 6$. Then*

$$Sz^*(G) \leq \begin{cases} (n^3+n^2-n-1)/4, & \text{if } n \text{ is odd},\\ (n^3+n^2-n)/4, & \text{if } n \text{ is even}.\end{cases}$$

with equality if and only if G is the graph obtained from the cycle C_{n-1} by duplicating a single vertex (see Fig. 1).

It is easy to see that for bicyclic graphs, the upper bound in Conjecture 1.1 is better than $\frac{n^2m}{4}$ for general graphs.

This paper is to give a confirmative proof to this conjecture.

2. Main results

For convenience, let B_n be the graph obtained from the cycle C_{n-1} by duplicating a single vertex (see Fig. 1). It is easy to check that

$$Sz^*(B_n) = \begin{cases} (n^3+n^2-n-1)/4, & \text{if } n \text{ is odd},\\ (n^3+n^2-n)/4, & \text{if } n \text{ is even}.\end{cases}$$

i.e., B_n satisfies the equality of Conjecture 1.1.

So, we are left to show that for any connected bicyclic graph G_n of order n, other than B_n, $Sz^*(G_n) < Sz^*(B_n)$. Using the fact that $n_u(e) + n_v(e) + n_0(e) = n$, we have

$$\begin{aligned} Sz^*(G) &= \sum_{e=uv\in E} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right)\\ &= \sum_{e=uv\in E} \left(\frac{n+n_u(e)-n_v(e)}{2}\right)\left(\frac{n-n_u(e)+n_v(e)}{2}\right)\\ &= \sum_{e=uv\in E} \frac{n^2-(n_u(e)-n_v(e))^2}{4}\\ &= \frac{mn^2}{4} - \frac{1}{4}\sum_{e=uv\in E}(n_u(e)-n_v(e))^2. \end{aligned}$$