Discrete Applied Mathematics 161 (2013) 2662–2677

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Digital circles, spheres and hyperspheres: From morphological models to analytical characterizations and topological properties

Jean-Luc Toutant [a,*], Eric Andres [b], Tristan Roussillon [c]

[a] Clermont Université, Université d'Auvergne, ISIT, UMR CNRS 6284, BP 10448, F-63000 Clermont-Ferrand, France
[b] Université de Poitiers, Laboratoire XLIM, SIC, UMR CNRS 7252, BP 30179, F-86962 Futuroscope Chasseneuil, France
[c] Université de Lyon, Université Lyon 2, LIRIS, UMR CNRS 5205, F-69676 Lyon, France

ARTICLE INFO

Article history:
Received 20 September 2011
Received in revised form 28 May 2013
Accepted 3 June 2013
Available online 29 June 2013

Keywords:
Digital geometry
Digital topology
Mathematical morphology
Digital circle and sphere
Analytical characterization

ABSTRACT

In this paper we provide an analytical description of various classes of digital circles, spheres and in some cases hyperspheres, defined in a morphological framework. The topological properties of these objects, especially the separation of the digital space, are discussed according to the shape of the structuring element. The proposed framework is generic enough so that it encompasses most of the digital circle definitions that appear in the literature and extends them to dimension 3 and sometimes dimension n.

1. Introduction

Digital circle generation, characterization and recognition have been important topics for many years in the digital geometry and pattern recognition communities. It is well known now that all digital straight lines are some sort of Reveillès digital straight line [32]. This arithmetical framework provides a way of defining digital hyperplanes too [5,32]. What is less well known is that there is not only one but many different types of digital circles in the literature. This is a problem when dealing with algorithms recognizing digital circles. Most recognition algorithms provide parameters of a Euclidean circle while the corresponding type of digital circle is implicit [9,11,14,20,30,34,35]. This makes comparison between different circle recognition algorithms dubious. Different sets may or may not be recognized as a digital circle by different algorithms. A second problem arises from the way a digital circle is defined. Digital circles are defined as the result of an algorithm or implicitly by a set of (topological) properties. A typical example is Bresenham's circle [8] which is either defined by its generation algorithm or topologically characterized as a 0-connected (8-connected in classical notation) digital approximation of a Euclidean circle of integer radius and integer coordinate center. This does not lead to a global mathematical definition of the object. Extensions to higher dimensions are thus complicated: a revealing fact is that there are almost no definition of digital spheres or hyperspheres in the literature [6,17].

In this paper, we propose a unified framework allowing to analytically characterize most of, if not all, known digital circles appearing in the literature [1,8,21,22,24,28,29,31]. Each of these digital circles is defined as the set of integer solutions of a system of analytical inequalities. Such a global mathematical definition provides natural extensions to the different types

* Corresponding author. Tel.: +33 471099072; fax: +33 4 71 09 90 49.
E-mail addresses: jean-luc.toutant@udamail.fr (J.-L. Toutant), eric.andres@univ-poitiers.fr (E. Andres), tristan.roussillon@liris.cnrs.fr (T. Roussillon).

0166-218X/$ – see front matter
http://dx.doi.org/10.1016/j.dam.2013.06.001

of digital circles, in particular, extension of the parameter domains and extension in dimensions. For instance, Bresenham's circle [8] can be easily extended to a digital circle that is not limited to integer radii or integer coordinate centers. It can also be extended to digital spheres or hyperspheres. This is a step forward compared to the results previously presented in [7].

In an n-dimensional Euclidean space, a sequence of morphological operations (dilations by a structuring element) and set-theoretic operations (intersection, union) is applied to a hypersurface $\mathcal{S}$ in order to define an offset region. The digitization of $\mathcal{S}$ is then the Gauss digitization of this offset region, i.e., the set of the integer coordinate points lying in it.

According to the type of structuring elements, two families of morphological digitization models are proposed. For both of them, the offset regions of a circle, a sphere and, in some cases, a hypersphere, are analytically described and the topological properties of their digitizations are studied.

For the first family of digitization models, the structuring elements correspond to norm-based balls. The norms we are considering are the Euclidean norm and the *adjacency norms* that encompass the ℓ^{∞}- and the ℓ^{1}-norms. The adjacency norms allow us to define k-separating digital hyperspheres. Analytical characterizations are provided for circles and spheres for all these norms. We propose only an analytical characterization of the hyperspheres for the Euclidean norm. Further work is needed for the general digital hypersphere analytical characterization for all adjacency norms.

The second family of digitization models is based on structuring elements, called *adjacency flakes*, that are subsets of the balls based on adjacency norms. The resulting digital hyperspheres are still k-separating, and even strictly k-separating (without any k-simple point) for one model. Besides they have much simpler analytical characterizations.

In Section 2, we introduce families (closed or semi-open and Gaussian or centered) of digitization models that are morphological in nature. Each model is parametrized by a structuring element. This allows to define different types of digital hyperspheres according to the shape of the structuring element.

In Sections 3 and 4, we propose digital hyperspheres based on balls of different norms. We recall some results for the digital hyperspheres based on the Euclidean norm [6] before introducing the *adjacency norms*. The adjacency balls enable us to define thin digital hyperspheres that separate $\mathbb{Z}^n$. We provide analytical characterizations only for circles and spheres. According to the adjacency norm considered, we define the Chebyshev and the Manhattan families. Supercover circles and spheres [10] are then closed centered Chebyshev circles and spheres. Bresenham's circles [8] are centered Manhattan circles.

Analytical characterizations for n-dimensional hyperspheres are proposed in Section 5, with a family of even thinner digital hyperspheres based on another kind of structuring elements. These structuring elements, called *adjacency flakes*, are specific subsets of the adjacency balls. The digital hyperspheres thus defined are compared with existing definitions in the literature, their topological properties are discussed and we provide their analytical characterization in any dimension.

Recalls and notations. Let $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ be the canonical basis of the n-dimensional Euclidean vector space. We denote by x_i the coordinate of a point or a vector $\mathbf{x}$ associated to $\mathbf{e_i}$. A *digital object* is a set of integer points. A *digital inequality* is an inequality with coefficients in $\mathbb{R}$ from which we retain only the integer coordinate solutions. A *digital analytical object* is a digital object defined by a finite set of digital inequalities.

For all $k \in \{0, \ldots, n-1\}$, two integer points $\mathbf{v}$ and $\mathbf{w}$ are said to be *k-adjacent* or *k-neighbors*, if for all $i \in \{1, \ldots, n\}$, $|v_i - w_i| \leq 1$ and $\sum_{j=1}^{n} |v_j - w_j| \leq n-k$. In the 2-dimensional plane, the 0- and 1-neighborhood notations correspond respectively to the classical 8- and 4-neighborhood notations. In the 3-dimensional space, the 0-, 1- and 2-neighborhood notations correspond respectively to the classical 26-,18- and 6-neighborhood notations.

A *k-path* is a sequence of integer points such that every two consecutive points in the sequence are k-adjacent. A digital object E is k-connected if there exists a k-path in E between any two points of E. A maximum k-connected subset of E is called a *k-connected component*. Let us suppose that the complement of a digital object E, $\mathbb{Z}^n \setminus \mathrm{E}$, admits exactly two k-connected components F_1 and F_2, or in other words that there exists no k-path joining integer points of F_1 and F_2, then E is said to be *k-separating* in $\mathbb{Z}^n$. If there is no path from F_1 to F_2 then E is said to be 0-separating or simply *separating*. A point $\mathbf{v}$ of a k-separating object E is said to be a *k-simple point* if $\mathrm{E} \setminus \{\mathbf{v}\}$ is still k-separating. A k-separating object that has no k-simple points is said to be *strictly k-separating*.

The logical *and* and *or* operators are denoted $\wedge$ and $\vee$ respectively.

Let $\oplus$ be the Minkowski addition, known as dilation, such that $\mathcal{A} \oplus \mathcal{B} = \cup_{\mathbf{b} \in \mathcal{B}} \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in \mathcal{A}\}$.

In the present paper, the focus is only on the n-dimensional hypersphere $\mathcal{S}_{\mathbf{c},r}$ of center $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and radius $r \in \mathbb{R}^+$ which is analytically defined as:

$$\mathcal{S}_{\mathbf{c},r} = \left\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : s_{\mathbf{c},r}(\mathbf{x}) = 0\right\}, \quad \text{with } s_{\mathbf{c},r}(\mathbf{x}) = \left(\sum_{i=1}^{n} (x_i - c_i)^2\right) - r^2.$$

We also introduce notations for the inside and the outside (strict or large) of such an hypersphere:

$$\mathcal{S}_{\mathbf{c},r}^{-\star} = \left\{\mathbf{x} \in \mathbb{R}^n : s_{\mathbf{c},r}(\mathbf{x}) < 0\right\}, \qquad \mathcal{S}_{\mathbf{c},r}^{+\star} = \left\{\mathbf{x} \in \mathbb{R}^n : s_{\mathbf{c},r}(\mathbf{x}) > 0\right\}, \qquad \mathcal{S}_{\mathbf{c},r}^{-} = \mathcal{S}_{\mathbf{c},r}^{-\star} \cup \mathcal{S}_{\mathbf{c},r} \quad \text{and} \quad \mathcal{S}_{\mathbf{c},r}^{+} = \mathcal{S}_{\mathbf{c},r}^{+\star} \cup \mathcal{S}_{\mathbf{c},r}.$$

2. Digitization models

Since the direct digitization of a hypersphere $\mathcal{S}_{\mathbf{c},r}$ has obviously not enough integer points to ensure good topological properties such as separation of the space, one first applies a sequence of morphological operations (dilations) to $\mathcal{S}_{\mathbf{c},r}$ in