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On a new type of distance Fibonacci numbers

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ABSTRACT

In this paper, we define a new kind of Fibonacci numbers generalized in the distance sense. This generalization is related to distance Fibonacci numbers and distance Lucas numbers, introduced quite recently. We also study distinct properties of these numbers for negative integers. Their representations and interpretations in graphs are also studied.

1. Introduction

Fibonacci numbers are given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial terms $F_0 = 0$ and $F_1 = 1$. There are many different generalizations in the theory of Fibonacci numbers, among other generalizations in the distance sense. In [6], Kwaśnik and Włoch introduced generalized Fibonacci numbers $F(k, n)$ defined recursively as follows: $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k + 1$ and $F(k, n) = n + 1$ for $n \leq k$. These numbers also have many interpretations in graphs; see [9–11]. The graph interpretation is closely related to the concept of k-independent sets in graphs; see [6]. Graph interpretations of the Fibonacci numbers give new tools for studying properties of numbers of the Fibonacci type. Quite recently, another type of distance generalization of Fibonacci numbers was introduced and studied in [1]. Distance Fibonacci numbers are denoted by $Fd(k, n)$ and defined recursively as follows: $Fd(k, n) = Fd(k, n-k+1)+Fd(k, n-k)$ for $n \geq k \geq 2$, with initial conditions $Fd(k, n) = 1$ for $0 \leq n \leq k-1$. Many interesting properties of these numbers were given in [1]. Moreover, two kinds of the cyclic version of distance Fibonacci numbers $Fd(k, n)$ were introduced in [2], namely distance Lucas numbers $L^{(1)}(k, n)$ and $L^{(2)}(k, n)$ of the first kind and the second kind, respectively.

In this paper, we introduce a new kind of distance Fibonacci numbers proceeding in the same direction. This paper is a sequel to the papers [1,2]. We also give combinatorial, graph, and matrix representations of distance Fibonacci numbers for negative integers. Moreover, we study different properties of these numbers in Pascal's triangle, too.

2. Generalized Fibonacci numbers and their combinatorial interpretations

In this section, we introduce a new generalization of Fibonacci numbers. Let $k \geq 1$, $n \geq 0$ be integers. By $(2, k)$-distance Fibonacci numbers $F_2(k, n)$, we mean generalized Fibonacci numbers defined recursively by the following relation:

$$F_2(k, n) = F_2(k, n-2) + F_2(k, n-k) \quad \text{for } n \geq k, \tag{1}$$

with initial conditions $F_2(k, i) = 1$ for $i = 0, 1, \ldots, k-1$.

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Table 1
$(2, k)$-distance Fibonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_2(1, n)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_2(2, n)$	1	1	2	2	4	4	8	8	16	16	32	32	64	64	128
$F_2(3, n)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37
$F_2(4, n)$	1	1	1	1	2	2	3	3	5	5	8	8	13	13	21
$F_2(5, n)$	1	1	1	1	1	2	2	3	3	4	5	6	8	9	12
$F_2(6, n)$	1	1	1	1	1	1	2	2	3	3	4	4	6	6	9

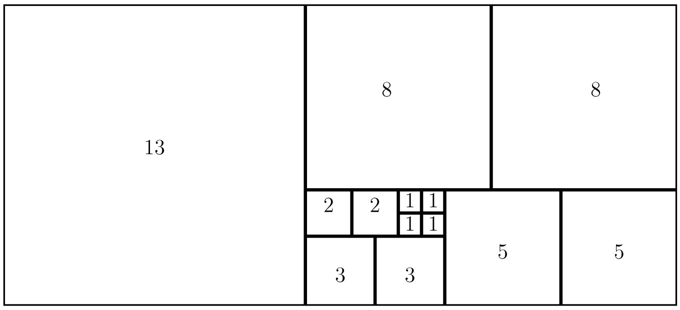

Fig. 1. A tiling interpretation of the double Fibonacci sequence $\{F_2(4, n)\}$.

From this definition, it immediately follows that, for even k, the terms of the sequence $\{F_2(k, n)\}$ satisfy the condition $F_2(k, 2n) = F_2(k, 2n+1)$. Consequently, for even k, we talk about "double" $(2, k)$-distance Fibonacci sequences; see Table 1. Table 1 includes the initial members of $(2, k)$-distance Fibonacci numbers for special values k and n.

Note that the sequence $F_2(1, n)$ is that of the classical Fibonacci numbers F_n. For $k = 2$, we obtain the known sequence with powers of 2 which double up. The number $F_2(2, n + 1)$ is the number of symmetric partitions of n or equivalently the number of subsets S of the set $\{1, 2, \ldots, n\}$ which satisfies the following condition: $m \in S$ implies that $n-m+1 \in S$; see [7]. If $k = 3$, then $\{F_2(3, n)\}$ is the well-known Padovan sequence, usually denoted by $\{Pv(n)\}$, and defined by the recurrence relation $Pv(n) = Pv(n-2) + Pv(n-3)$ for $n \geq 3$ with $Pv(0) = Pv(1) = Pv(2) = 1$. It is worth mentioning that Padovan numbers have some interesting applications in graphs; among others, $Pv(n-3)$ is the maximum value of the number of all maximal independent sets including the set of pendant vertices among all n-vertex trees; see [8]. If $k = 4$, then the sequence of numbers $F_2(4, n)$ is the known double Fibonacci sequence; for its special interpretations, see [7].

Fig. 1 gives a tiling interpretation of the double Fibonacci sequence $\{F_2(4, n)\}$ with squares whose sides are successive Fibonacci numbers in length. Note that the classical Fibonacci sequence has a similar interpretation and it is used for building logarithmic spirals.

Now, we present some combinatorial interpretations of $(2, k)$-distance Fibonacci numbers. The classical Fibonacci numbers i.e. numbers $F_2(1, n)$, have many combinatorial interpretations, for instance, the interpretation which is related to special set decomposition; see [1]. We recall it.

Let $X = \{1, 2, \ldots, n\}$, $n \geq 1$ be the set of n integers. Let $\mathcal{Y}^* = \{Y_t^*; t \in T\}$ be a family of disjoint subsets of the set X such that subsets Y_t^* satisfy the following conditions:

(a) $|Y_t^*| \in \{1, 2\}$,
(b) if $|Y_t^*| = 2$, then Y_t^* contains two consecutive integers,
(c) $X \setminus \bigcup_{t\in T} Y_t^* = \emptyset$.

The family $\mathcal{Y}^*$ is a decomposition of the set X of subsets having one or two elements, such that all two-element subsets contain consecutive integers. It is well known that the number of all families $\mathcal{Y}^*$ is equal to the Fibonacci number F_n.

If $k = 2$, then we obtain a previously mentioned sequence with powers of 2 which double up, and for many cases we have to exclude this sequence from our considerations.

Assuming $k \geq 3$, and using the idea of combinatorial interpretation for classical Fibonacci numbers presented earlier, we can give an interpretation of $(2, k)$-distance Fibonacci numbers.

Let $k \geq 3$, $n \geq 2$ be integers. Assume that $X = \{1, 2, \ldots, n\}$ is the set of n integers, and let $\mathcal{Y} = \{Y_t; t \in T\}$ be the family of disjoint subsets of the set X such that each subset Y_t, $t \in T$ contains consecutive integers and satisfies the following conditions:

(d) $|Y_t| \in \{2, k\}$ for $t \in T$,
(e) $|X \setminus \bigcup_{t\in T} Y_t| \in \{0, 1\}$,
(f) if $m \in (X \setminus \bigcup_{t\in T} Y_t)$, then $m = n$.