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Generalized degeneracy, dynamic monopolies and maximum degenerate subgraphs

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ABSTRACT

A graph G is said to be k-degenerate if any subgraph of G contains a vertex of degree at most k. The degeneracy of graphs has many applications and was widely studied in graph theory. We first generalize k-degeneracy by introducing κ-degeneracy of graphs, where κ is any non-negative function on the vertex set of the graph. We present a polynomial time algorithm to determine whether a graph is κ-degenerate. Let $\tau : V(G) \to \mathbb{Z}$ be an assignment of thresholds to the vertices of G. A subset of vertices D is said to be a τ-dynamic monopoly of G, if the vertices of G can be partitioned into subsets $D_0, D_1, \ldots, D_k$ such that $D_0 = D$ and for any $i \in \{0, \ldots, k-1\}$, each vertex v in D_{i+1} has at least $\tau(v)$ neighbors in $D_0 \cup \cdots \cup D_i$. The concept of dynamic monopolies is used for the formulation and analysis of spread of influence such as disease or opinion in social networks and is the subject of active research in recent years. We obtain a relationship between degeneracy and dynamic monopoly of graphs and show that these two concepts are dual of each other. Using this relationship, we introduce and study $dyn_t(G)$, which is the smallest cardinality of any τ-dynamic monopoly among all threshold assignments τ with average threshold $\bar{\tau} = t$. We give an explicit formula for $dyn_t(G)$, and obtain some lower and upper bounds for it. We show that $dyn_t(G)$ is NP-complete but for complete multipartite graphs and some other classes of graphs it can be solved by polynomial time algorithms. For the regular graphs, $dyn_t(G)$ can be approximated within a ratio of nearly 2. Finally we consider the problem of determining the maximum size of κ-degenerate (or k-degenerate) induced subgraphs in any graph and obtain some upper and lower bounds for the maximum size of such subgraphs.

1. Introduction

All graphs in this paper are finite, undirected and simple. For standard graph theoretical notions and notations we refer the reader to [6]. Let k be any non-negative integer. A graph G is said to be a k-degenerate graph if any subgraph of G contains a vertex of degree at most k. It is a well-known fact that G is k-degenerate if and only if the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that the degree of v_i in the subgraph of G induced by $\{v_1, \ldots, v_i\}$ is at most k for any $i \in \{1, \ldots, n\}$. The concept of degeneracy has many applications in graph theory, for example in algorithmic, extremal and chromatic graph theory and has caused many interesting results and problems in these areas (see e.g. [14]).

We generalize the concept of k-degeneracy as follows. Let κ be any assignment of non-negative integers to the vertices of G. We say a graph G is κ-degenerate if the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that the degree of v_i in the subgraph of G induced by $\{v_1, \ldots, v_i\}$ is at most $\kappa(v_i)$ for any $i \in \{1, \ldots, n\}$. Note that when κ is a constant function then κ-degeneracy is equivalent to k-degeneracy. Throughout this paper by $\mathbb{N}$ we mean the set of non-negative integers.

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The other concept to be discussed in this paper is irreversible dynamic monopoly (or simply dynamic monopoly). In recent years, great attention has been paid to the modeling and analysis of the spread of belief or influence in social networks. The concept of dynamic monopolies was introduced in order to formulate these problems. By a threshold assignment for a graph G we mean any function $\tau : V(G) \to \mathbb{Z}$ such that $\tau(v) \leq \deg(v)$ for any vertex v, where $V(G)$ is the vertex set of G and $\deg(v)$ is the degree of v in G. A subset $D \subseteq V(G)$ is called a dynamic monopoly (or a τ-dynamic monopoly) if there exists a partition of $V(G)$ into subsets $D_0, D_1, \ldots, D_k$ such that $D_0 = D$ and for any $i = 1, \ldots, k-1$ each vertex v in D_{i+1} has at least $\tau(v)$ neighbors in $D_0 \cup \cdots \cup D_i$. When a vertex v belongs to D_i for some i, we say v is activated at time step i. Denote the smallest cardinality of any τ-dynamic monopoly of G by $\mathrm{dyn}_\tau(G)$. Note that we have extended the notion of dynamic monopolies by allowing the vertices of the graph to have non-positive thresholds. In the standard definition of dynamic monopolies all thresholds are non-negative integers. When a vertex v has threshold $\tau(v) \leq 0$ then v is automatically an active vertex and it belongs to the monopoly; but in the definition of the size of a dynamic monopoly such vertices are not counted. In precise words, by the size of a dynamic monopoly $D = D_0, D_1, \ldots, D_k$ we mean $|D \setminus \{v : \tau(v) \leq 0\}|$. A τ-dynamic monopoly is also called k-conversion set whenever τ is the constant function k [9]. Also dynamic monopolies were studied under the terminology of target set selection (e.g. [1]). Dynamic monopolies (including target set selection problem and k-conversion sets) have been widely studied with various types of threshold assignments [1–3,7–10,15,18]. Random subsets of a graph as dynamic monopolies were studied in [16]. This setup with constant thresholds is also studied under the name of bootstrap percolation (see e.g. [5]). For more related works on dynamic monopolies we refer the reader to [18], where dynamic monopolies with general and probabilistic threshold assignments were considered. As we mentioned before, we have extended the notion of dynamic monopolies by allowing the vertices of the graph to have non-positive thresholds in order to obtain more applications of dynamic monopolies. One of these applications is the relationship of dynamic monopolies with κ-degeneracy to be explored in this paper and in finding k-degenerate induced subgraphs with maximum cardinality. Some results of this paper were already reported in arXiv by the author [19].

The outline of the paper is as follows. In the next section an $\mathcal{O}(n^2)$ algorithm is given for deciding whether a graph is κ-degenerate (Theorem 1). Then it is proved that subject to some conditions, a subset M of the vertices of G is a τ-dynamic monopoly if and only if its complement is κ-degenerate (Theorem 3). In Section 2 we define and study $\mathrm{dyn}_{\bar{\tau}=t}(G)$. We show that $\mathrm{dyn}_{\bar{\tau}=t}(G)$ is expressed in terms of the maximum order of any induced subgraph with no more than a certain number of edges (Theorem 4). First, we show that $\mathrm{dyn}_{\bar{\tau}=t}(G)$ is NP-complete (Theorem 6). But we show that $\mathrm{dyn}_{\bar{\tau}=t}(G)$ can be obtained by a polynomial time algorithm for complete multipartite graphs G (Theorem 7) and also for some families of graphs with special values of t (Theorem 9). Then we obtain a lower bound (Corollary 1) and an upper bound (Theorem 10) for $\mathrm{dyn}_{\bar{\tau}=t}(G)$ in general graphs and for regular graphs (Corollary 2). Theorem 11 shows that $\mathrm{dyn}_{\bar{\tau}=t}(G)$ can be approximated within a ratio of nearly 2 in case of regular graphs. In the last section we consider the maximum order of any κ-degenerate induced subgraph in a graph and review some related results.

2. κ-degeneracy and dynamic monopolies

Let κ be any assignment of non-negative integers to the vertices of G. The graph G is κ-degenerate if the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that the degree of v_i in the subgraph of G induced by $\{v_1, \ldots, v_i\}$ is at most $\kappa(v_i)$ for any $i \in \{1, \ldots, n\}$. Since some of our results are in terms of κ-degeneracy, it is necessary to present a method to check whether a graph is κ-degenerate. We first need the following proposition. Note that if G is κ-degenerate then $|E(G)| \leq \sum_{v \in G} \kappa(v)$.

Proposition 1. (i) *Assume that G is κ-degenerate. Then there exists a vertex v in G such that $0 \leq \kappa(v) - \deg(v) \leq \sum_{v \in V(G)} \kappa(v) - |E(G)|$.*

(ii) *Let $\kappa : V(G) \to \mathbb{N}$ be given. Let the vertex v of G be such that $0 \leq \kappa(v) - \deg(v) \leq \sum_{v \in V(G)} \kappa(v) - |E(G)|$. Then G is κ-degenerate if and only if $G \setminus \{v\}$ is κ'-degenerate, where κ' is obtained by restricting κ to $V(G) \setminus \{v\}$.*

Proof. To prove (i), since G is κ-degenerate then there exist $v_1, \ldots, v_n$ such that $\deg_{G[v_1,\ldots,v_i]}(v_i) \leq \kappa(v_i)$, for any i. We have $|E(G)| = \sum_i \deg_{G[v_1,\ldots,v_i]}(v_i) \leq \sum_{v \in V(G)} \kappa(v)$. Obviously $G' = G \setminus \{v_n\}$ is κ'-degenerate. So we write the same inequality for the edges of G'. We obtain $|E(G)| - \deg_G(v_n) \leq \sum_{i \in \{1,\ldots,n-1\}} \kappa(v_i)$. Hence v_n satisfies the condition of part (i).

To prove (ii) we note that if $G \setminus \{v\}$ is κ-degenerate by the ordering $v_1, v_2, \ldots, v_{n-1}$, then G is κ-degenerate by the ordering $v_1, \ldots, v_{n-1}, v$. The converse is trivial since if a graph is κ-degenerate then any subgraph of it is also κ-degenerate. □

Based on Proposition 1 we obtain an algorithm which decides if a graph G is κ-degenerate.

Theorem 1. *There exists an $\mathcal{O}(n^2)$ algorithm such that given a graph G on n vertices and a function κ, it determines whether G is κ-degenerate.*

Proof. While $|E(G)| < \sum_{v \in V(G)} \kappa(v)$ we seek for a vertex v such that $0 \leq \kappa(v) - \deg(v) \leq \sum_{v \in V(G)} \kappa(v) - |E(G)|$. If there exists no such vertex then G is not κ-degenerate by Proposition 1. If there exists such a vertex v_1 we replace G by $G_1 = G \setminus \{v_1\}$ and replace κ by κ_1, where κ_1 is obtained by the restriction of κ to $V(G) \setminus v_1$. While $|E(G_i)| < \sum_{v \in V(G_i)} \kappa_i(v)$ we repeat the same procedure and seek for a vertex v with $0 \leq \kappa_i(v) - \deg_{G_i}(v) \leq \sum_{v \in V(G_i)} \kappa_i(v) - |E(G_i)|$. If at some step there exists no such vertex then the algorithm answers "NO". If there exists such a vertex then we do the same procedure as before.