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Note

On the vertex-pancyclicity of hypertournaments

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ABSTRACT

A k-hypertournament H on n vertices, where $2 \leq k \leq n$, is a pair $H = (V, A_H)$, where V is the vertex set of H and A_H is a set of k-tuples of vertices, called arcs, such that, for all subsets $S \subseteq V$ with $|S| = k$, A_H contains exactly one permutation of S as an arc. Gutin and Yeo (1997) showed in [2] that any strong k-hypertournament H on n vertices, where $3 \leq k \leq n-2$, is Hamiltonian, and posed the question as to whether the result could be extended to vertex-pancyclicity. As a response, Petrovic and Thomassen (2006) in [4] and Yang (2009) in [6] gave some sufficient conditions for a strong hypertournament to be vertex-pancyclic.

In this paper, we prove that, if H is a strong k-hypertournament on n vertices, where $3 \leq k \leq n-2$, then H is vertex-pancyclic. This extends the aforementioned results and Moon's theorem for tournaments. Furthermore, our result is best possible in the sense that the bound $k \leq n-2$ is tight.

1. Introduction and terminology

A *k-hypertournament H on n vertices*, where $2 \leq k \leq n$, is a pair $H = (V, A_H)$, where V is the vertex set of H and A_H is a set of k-tuples of vertices, called arcs, such that, for all subsets $S \subseteq V$ with $|S| = k$, A_H contains exactly one permutation of S as an arc. A tournament is a 2-hypertournament.

Let $H = (V, A_H)$ be a k-hypertournament on n vertices. For a pair $x, y \in V$ of distinct vertices, $A_H(x, y) \subseteq A_H$ denotes the set of arcs in which x precedes y. A *(v_1, v_{l+1})-path of length l*, also called *l-path from v_1 to v_{l+1}*, in H is a sequence $v_1a_1v_2a_2 \ldots v_la_lv_{l+1}$, where $v_1, \ldots, v_{l+1} \in V$ are pairwise distinct vertices, $a_1, \ldots, a_l \in A_H$ are pairwise distinct arcs, and $a_i \in A_H(v_i, v_{i+1})$ holds for all $1 \leq i \leq l$. $V(P)$ denotes the set of vertices contained in a path P. An *l-cycle in H* is defined analogously, with the only distinction that we have $v_1 = v_{l+1}$. When considering an l-cycle $v_1a_1v_2a_2 \ldots v_la_lv_1$ in H, we will define v_{l+1} as v_1 implicitly for convenience. An n-cycle ($(n-1)$-path, respectively) in H is called *Hamiltonian*, and the hypertournament H is called *Hamiltonian*, if it contains a Hamiltonian cycle. H is *strong*, if it contains an (x, y)-path for all distinct vertices $x, y \in V$. A vertex $x \in V$ is called *pancyclic*, if it is contained in an l-cycle for all $l \in \{3, \ldots, n\}$. The hypertournament H is called *vertex-pancyclic*, if all its vertices are pancyclic.

A *semicomplete digraph* D is a pair $D = (V, A_D)$, where V is the vertex set of D and the arc set A_D of D contains at least one of the arcs xy and yx for all distinct vertices $x, y \in V$.

The definitions of paths, cycles, etc. for semicomplete digraphs are analogous to those for hypertournaments, but we omit the notation of arcs in paths and cycles, since the connected vertices already determine the arc connecting them. Furthermore, a *strong component* of a semicomplete digraph D is a maximal strong subdigraph of D. The strong components $D_1, \ldots, D_s$ of a semicomplete digraph $D = (V, A_D)$ can be ordered in such a way that we have $xy \in A_D$ for all $x \in D_i$ and $y \in D_j$ with $1 \leq i < j \leq s$. This order is called the *strong decomposition of D*.

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Tournaments are the best-studied class of digraphs. Their strong structure permits a multitude of significant propositions and characterizations of tournaments with desirable properties. Consider for example Rédei's theorem and Camion's theorem, respectively.

Theorem 1.1 ([5]). *Every tournament contains a Hamiltonian path.*

Theorem 1.2 ([1]). *Every strong tournament contains a Hamiltonian cycle.*

In an effort to find classes containing more digraphs than tournaments which the classical results for tournaments still extend to, there have been many generalizations of tournaments, for example locally semicomplete digraphs, multipartite tournaments, or hypertournaments, to name a few. But the loosened structure of these digraphs can, at times, cause the proofs of generalized basic results for tournaments to become difficult if not entirely impossible. Theorem 1.2, for example, does not hold for multipartite tournaments. To prove this theorem for hypertournaments, Gutin and Yeo introduced the majority digraph of a hypertournament in 1997 as follows.

For a k-hypertournament $H = (V, A_H)$ on n vertices, the *majority digraph* $M(H) = (V, A_{\text{maj}}(H))$ *of* H is a digraph on the same vertex set, and for a pair $x, y \in V$ of distinct vertices, xy is in $A_{\text{maj}}(H)$ iff $|A_H(x, y)| \geq |A_H(y, x)|$, which is equivalent to

$$|A_H(x, y)| \geq \frac{1}{2}\binom{n-2}{k-2}.$$

By definition, there is an arc between every pair of distinct vertices; thus $M(H)$ is a semicomplete digraph.

Using this substructure, Gutin and Yeo were able to give the following generalizations of Theorems 1.1 and 1.2, respectively.

Theorem 1.3 ([2]). *Every k-hypertournament on $n > k \geq 2$ vertices contains a Hamiltonian path.*

Theorem 1.4 ([2]). *Every strong k-hypertournament on n vertices, where $3 \leq k \leq n-2$, contains a Hamiltonian cycle.*

Furthermore, they gave an example of a strong $(n-1)$-hypertournament on n vertices which does not contain a Hamiltonian cycle. Since an n-hypertournament on n vertices contains only one arc, the bound in Theorem 1.4 is tight. Their question as to whether the result could be extended to vertex-pancyclicity was considered by Petrovic and Thomassen in 2006. They gave the following sufficient conditions.

Theorem 1.5 ([4]). *Let H be a strong k-hypertournament on n vertices. If $k = 3$ and $n \geq 32$ or $k \geq 4$ and $n \geq k + 25$, then H is vertex-pancyclic.*

In 2009, the given bounds were improved by Yang.

Theorem 1.6 ([6]). *Let H be a strong k-hypertournament on n vertices. If $k = 3$ and $n \geq 15$, $k = 4$ and $n \geq 11$, $k \geq 5$ and $n \geq k + 4$ or $k \geq 8$ and $n \geq k + 3$, then H is vertex-pancyclic.*

In the next section, we will extend Moon's theorem to all strong k-hypertournaments on n vertices, where $3 \leq k \leq n-2$, which is obviously best possible, since the bound is best possible in Theorem 1.4.

Theorem 1.7 ([3]). *Every strong tournament is vertex-pancyclic.*

Remark 1.8. Theorem 1.7 is also valid for semicomplete digraphs.

But first we account for the case that the majority digraph of a strong hypertournament is not itself strong by giving the following definition.

If H is a strong hypertournament but $M(H)$ is not strong, then there is a path $P = y_1 a_1 y_2 \dots a_l y_{l+1}$ in H from the terminal component of the strong decomposition of $M(H)$ to the initial component of shortest length. We call such a path P a *strengthening path of* $M(H)$ *in* H, and define the *corresponding strengthened majority digraph*

$$M(H, P) := (V, A^P_{\text{maj}}(H)) \quad \text{through } A^P_{\text{maj}}(H) := A_{\text{maj}}(H) \cup \{y_i y_{i+1} \mid 1 \leq i \leq l\}.$$

For $i \in \{1, \dots, l\}$, we call $a_{y_i y_{i+1}} := a_i \in A_H(y_i, y_{i+1})$ the *arc corresponding to* $y_i y_{i+1}$ *in* P. Obviously, $M(H, P)$ is a strong semicomplete digraph.

2. Main result

Theorem 2.1. *Every strong k-hypertournament $H = (V, A_H)$ on n vertices, where $3 \leq k \leq n-2$, is vertex-pancyclic.*

Proof. H is Hamiltonian by Theorem 1.4. Thus, all we need to show is that every vertex of H is contained in an l-cycle for all $l \in \{3, \dots, n-1\}$. Let $v \in V$ and $l \in \{3, \dots, n-1\}$ be chosen arbitrarily. If $M(H)$ is strong, then it is vertex-pancyclic, by