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Polyhedral results and exact algorithms for the asymmetric travelling salesman problem with replenishment arcs

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Abstract

The asymmetric travelling salesman problem with replenishment arcs (RATSP), arising from work related to aircraft routing, is a generalisation of the well-known ATSP. In this paper, we introduce a polynomial size mixed-integer linear programming (MILP) formulation for the RATSP, and improve an existing exponential size ILP formulation of Zhu [The aircraft rotation problem, Ph.D. Thesis, Georgia Institute of Technology, Atlanta, 1994] by proposing two classes of stronger cuts. We present results that under certain conditions, these two classes of stronger cuts are facet-defining for the RATS polytope, and that ATSP facets can be lifted, to give RATSP facets. We implement our polyhedral findings and develop a Lagrangean relaxation (LR)-based branch-and-bound (BNB) algorithm for the RATSP, and compare this method with solving the polynomial size formulation using ILOG Cplex 9.0, using both randomly generated problems and aircraft routing problems. Finally we compare our methods with the existing method of Boland et al. [The asymmetric traveling salesman problem with replenishment arcs, European J. Oper. Res. 123 (2000) 408–427]. It turns out that both of our methods are much faster than that of Boland et al. [The asymmetric traveling salesman problem with replenishment arcs, European J. Oper. Res. 123 (2000) 408–427], and that the LR-based BNB method is more efficient for problems that resemble the aircraft rotation problems.

Keywords: Combinatorial optimisation; Travelling salesman; Polyhedral analysis; Lagrangean relaxation; Branch-and-bound method

1. Introduction

The asymmetric travelling salesman problem with replenishment arcs (RATSP) is a generalisation of the well-known asymmetric travelling salesman problem (ATSP). The problem was introduced by Boland et al. in [6] and arose in the context of airline planning (see, for example, [5]). Given a digraph $G=(V, A)$ with node set V, arc set A, and asymmetric costs on the arcs $c \in \mathbb{R}^{|A|}$, a tour in G is defined to be a sequence that starts from a node, visits each other node exactly once, then finishes at the node where the tour is started. A solution of the RATSP, like that of the ATSP, induces a tour in G which minimizes the total cost. However, the tour must satisfy additional constraints: the arc set A is partitioned into *replenishment arcs*, denoted by $\mathscr{R}$ and *ordinary arcs*, denoted by $\mathscr{Q}=A\backslash\mathscr{R}$, each node $i \in V$ has a positive weight w_i associated with it, and there is a positive weight limit W; a feasible tour cannot accumulate more than W units of weight before using a replenishment arc. We restrict the RATSP to have at least one replenishment arc, i.e. to have

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$\mathscr{R} \neq \emptyset$, and to have the sum of weights of all nodes greater than the weight limit, i.e. to have $\sum_{i \in V} w_i > W$; otherwise the problem reverts to the usual ATSP. The RATSP can be represented by a weighted digraph, as defined below.

Definition 1.1. A weighted digraph $\mathscr{G}=(V, A, W, w)$ is a directed graph with node set V, arc set A partitioned according to $A = \mathscr{Q} \cup \mathscr{R}$, for $\mathscr{Q}$ the set of ordinary arcs and $\mathscr{R}$ the set of replenishment arcs, with $\mathscr{Q} \cap \mathscr{R} = \emptyset$, weights on nodes $w \in \mathscr{Z}_+^{|V|}$, weight limit $W \in \mathscr{Z}_+$, where $W \geqslant w_i$ for all $i \in V$, and $W \geqslant w_i + w_j$ for all $(i, j) \in \mathscr{Q}$.

Note that Definition 1.1 does not restrict $\mathscr{Q}$, since no ordinary arc $(i, j) \in \mathscr{Q}$ with $w_i + w_j > W$ can possibly be used in any feasible solution.

RATSP can be used to model the aircraft rotation problems (ARP) discussed in [5]. The ARP is to sequence a set of flight legs, subject to the satisfaction of maintenance requirements, and to maximize the profit from linking the flight legs. In the RATSP model, nodes represent flights, arcs represent aircraft connections, weights represent flying times and replenishment arcs indicate connections occurring at a maintenance port with sufficient time (or facility) available to perform maintenance. Aside from the ARP, the RATSP could also be used to model some forms of the black and white travelling salesman problem (BWTSP) (see, for example, Bourgeois et al. [7] and Ghiani et al. [25]), and some forms of the asymmetric capacitated vehicle routing problem (ACVRP) (see, for example [40]). In fact, the latter can be viewed as a special case of the RATSP if the fleet size constraint is absent. In this paper, we are interested in solving problems that naturally take the form of an RATSP, and in developing an exact algorithm that exploits its natural structure.

1.1. Previous work and related research

There has been little previous work on the RATSP. Boland et al. in [6] present a formulation with an exponential number of variables and constraints, and propose a branch-and-price-and-cut method. They experimented on randomly generated RATSPs very similar to those we report on in Section 4, and some ARPs with up to 190 nodes and 6244 arcs. The RATSP instances tested has only 36 nodes. The method of [6] solved problems with low arc density, or with a high proportion of replenishment arcs, however solution times are often long considering the size of the problems. For all problems solved to optimality, the number of branch-and-bound (BNB) nodes required is small, however the time taken to solve the linear programs with the cut and column generation method is large, since the column generation subproblems are themselves NP hard. This motivated our Lagrangean relaxation (LR)-based BNB algorithm. The LR subproblems are assignment problems (AP) that can be solved in $\mathscr{O}(n^2)$ time and that returns natural integer solutions, and because of this, the separation of the rest of constraints becomes trivial. In our experiments with the LR-based BNB algorithm, we solved much larger problems: RATSP instances with up to 100 nodes, and ARPs with up to 519 nodes and up to 42 732 arcs. Our methods are particularly efficient when solving problems that were considered difficult in [6]: those with high arc density and/or low proportion of replenishment arcs.

In heuristic methods, Mak and Boland [33] proposed: (1) a simulated annealing (SA) algorithm and (2) a LR heuristics. The Lagrangean dual problems are solved by a truncated subgradient optimisation method. These methods were tested in [33] on problems similar to those used in [6], and were found to take much less time to yield upper and lower bounds with a gap of less than 3%. The SA algorithm performed particularly well: it found optimal solutions in a third of the problem instances tested, and gaps were below 2.4% in all instances.

In the context of the ARP, Zhu [42] (see also [11]) develops an integer linear programming (ILP) formulation based on the DFJ-formulation [13] for the ATSP and presents three solution techniques: (1) an LR heuristic with bounds obtained by solving a LD problem with a scaled, partial subgradient optimisation method; (2) a BNB algorithm using bounds obtained by solving an AP relaxation at each node of the BNB tree; and (3) a local search heuristic [42] only tested 11 ARPs. The work of Barnhart et al. in [5] on ARPs does not add anything to the conclusions of Boland et al. in [6] with respect to the RATSP, but does provide insight into the performance of methods on problems having aircraft rotation special structure.

There is of course a great deal of work available on closely related problems, such as the ATSP. It would appear that the most computationally successful methods for solving the ATSP to date are branch-and-cut (BNC) method of [20] in the case that the AP relaxation does not provide a tight bound on the problem, and the AP relaxation based BNB method of [9], otherwise. With the ATSP, it is well known that the separation of the subtour elimination constraint, commonly referred to as SEC, (see [13]), with fractional solutions, can be performed in polynomial time. For the RATSPs, however,