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The Sheffer group and the Riordan group

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Abstract

We define the Sheffer group of all Sheffer-type polynomials and prove the isomorphism between the Sheffer group and the Riordan group. An equivalence of the Riordan array pair and generalized Stirling number pair is also presented. Finally, we discuss a higher dimensional extension of Riordan array pairs.

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1. Introduction

Riordan array is a special type of infinite lower-triangular matrix and the set of all Riordan matrices forms a group called the Riordan group, which was first defined in 1991 by Shapiro et al. [23]. Some of main results on the Riordan group and its applications to the combinatorial sums and identities can be found in [15,22,23,26]. In particular, in the work by Sprugnoli (cf. [24,25]). In this paper, we will define an operation on the set of all Sheffer polynomial sequences so it forms a group called as the Sheffer group, which gives a general pattern consisting of various special Sheffer-type polynomial sequences as elements. We will show that every element of the group and its inverse are the potential polynomials of a pair of generalized Stirling numbers (GSNs) (see 3.7), and the isomorphism between the Sheffer group and the Riordan group (see 2.2). Hence, the established results on the Sheffer group connect the Riordan group, GSN pairs, and Riordan arrays, which can lead a comprehensive study on all of the topics. For instance, the Sheffer group and the related GSN-pairs and their inverse relations can be used to derive combinatorial identities as well as algebraic identities containing the Sheffer-type polynomials.

As what mentioned in [15] (see also in [1]), "The concept of representing columns of infinite matrices by formal power series is not new and goes back to Schur's paper on Faber polynomials in 1945 (cf. [21])". A formal power series

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in auxiliary variable t of the form

$$b(t) = b_0 + b_1 t + b_2 t^2 + \cdots = \sum_{n \geqslant 0} b_n t^n$$

is called an ordinary generating function of the sequence $\{b_n\}$.

Definition 1.1. Let $A(t)$ and $g(t)$ be any given formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $A(0) = 1$, $g(0) = 0$ and $g'(0) \neq 0$. We call the infinite matrix $D = [d_{n,k}]_{n,k \geqslant 0}$ with real entries or complex entries a generalized Riordan matrix (the originally defined Riordan matrices need $g'(0) = 1$) if its kth column satisfies

$$\sum_{n \geqslant 0} d_{n,k} t^n = A(t)(g(t))^k; \tag{1.1}$$

that is,

$$d_{n,k} = [t^n] A(t)(g(t))^k.$$

The Riordan matrix is denoted by $[d_{n,k}]$ or $(A(t), f(t))$.

Example 1.1. Riordan matrices $(1, t)$ and $(1/(1-t), t/(1-t))$ are the identity matrix and Pascal's triangle, respectively.
If $(A(t), g(t)$ and $(B(t), f(t))$ are Riordan matrices, then

$$(A(t), g(t)) * (B(t), f(t)) := (A(t)B(g(t)), f(g(t))) \tag{1.2}$$

is called the matrix multiplication, i.e., for $(A(t), g(t)) = [d_{nk}]_{n \geqslant k \geqslant 0}$ and $(B(t), f(t)) = [c_{nk}]_{n \geqslant k \geqslant 0}$ we have

$$(A(t), g(t)) * (B(t), f(t)) := (A(t)B(g(t)) \cdot f(g(t))) = [d_{nk}][c_{nk}]. \tag{1.3}$$

The set of all Riordan matrices is a group under the matrix multiplication (cf. [23–25]).

Definition 1.2. Let $A(t)$ and $g(t)$ be defined as 1.1. Then the polynomials $p_n(x)$ $(n = 0, 1, 2, \ldots)$ defined by the generating function (GF)

$$A(t)e^{xg(t)} = \sum_{n \geqslant 0} p_n(x) t^n \tag{1.4}$$

are called Sheffer-type polynomials with $p_0(x) = 1$. Accordingly, $p_n(D)$ with $D \equiv \mathrm{d}/\mathrm{d}t$ is called Sheffer-type differential operator of degree n associated with $A(t)$ and $g(t)$. In particular, $p_0(D) \equiv I$ is the identity operator.

The set of all Sheffer-type polynomial sequences $\{p_n(x) = [t^n]A(t)e^{xg(t)}\}$ with an operation, "umbral composition" (cf. [18,19]), shown later forms a group called the Sheffer group. We will also show that the Riordan group and the Sheffer group are isomorphic.

In Roman's book [18], $\{S_n = n! p_n(x)\}$ is called Sheffer sequence (also cf. [19,20]). Certain recurrence relation of $p_n(x)$ can be found in Hsu–Shiue's paper [12]. There are two special kinds of weighted Stirling numbers defined by Carlitz [4] (see also [2,8]). We now give the following definition of the generalized Stirling numbers.

Definition 1.3. Let $A(t)$ and $g(t)$ be defined as 1.1, and let

$$\frac{1}{k!} A(t)(g(t))^k = \sum_{n \geqslant k} \sigma(n, k) \frac{t^n}{n!}. \tag{1.5}$$

Then $\sigma(n, k)$ is called the generalized Stirling number with respect to $A(t)$ and $g(t)$.

The special case of $A(t) \equiv 1$ was studied in [10].