Discrete Applied Mathematics 159 (2011) 939–944

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

An axiomatization of the median procedure on the n-cube

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ARTICLE INFO

Article history:
Received 9 May 2010
Received in revised form 4 February 2011
Accepted 8 February 2011
Available online 9 March 2011

Keywords:
Median
Median function
Hypercube
Location function
Consensus axiom

ABSTRACT

The general problem in location theory deals with functions that find sites to minimize some cost, or maximize some benefit, to a given set of clients. In the discrete case sites and clients are represented by vertices of a graph, in the continuous case by points of a network. The axiomatic approach seeks to uniquely distinguish certain specific location functions among all the arbitrary functions that address this problem by using a list of intuitively pleasing axioms. The median function minimizes the sum of the distances to the client locations. This function satisfies three simple and natural axioms: anonymity, betweenness, and consistency. They suffice on tree networks (continuous case) as shown by Vohra (1996) [19], and on cube-free median graphs (discrete case) as shown by McMorris et al. (1998) [9]. In the latter paper, in the case of arbitrary median graphs, a fourth axiom was added to characterize the median function. In this note we show that the above three natural axioms still suffice for the hypercubes, a special instance of arbitrary median graphs.

1. Introduction

Facility location problems involve a set of 'clients' at various given positions. One seeks a set of positions acceptable for the providing of a given service. Graphs and networks are natural for this type of problem since they can model a network of roads. Indeed, hundreds of papers have been written about location problems on graphs and networks using the geodesic metric, see for example the reference lists in [12,16]. The specific application dictates which objective function might be appropriate. To locate a site for an emergency service, one might seek to minimize the greatest distance to any client: hence the center is a good choice. For a facility designed for the delivery of goods, the median set is reasonable. Many versions of 'central' subgraphs have been considered on various classes of graphs, see [5,21,22,18,17].

In social choice theory, voters or clients provide a list of preferences for the outcomes of a decision procedure. One seeks a 'consensus', namely a set of outcomes that best satisfy the voters' preferences. See the list of references in [20] for surveys of social choice functions.

In both settings, that of consensus and that of location, numerous researchers have addressed the issue of identifying an objective function via a succinct 'wish list' of desired properties. The goal here is to identify functions for which this list, or something close, gives a characterization. This method allows one to argue in favor of a particular set of locations (or particular consensus) as being precisely that satisfying certain desirable properties. Another perspective is that one requires that consensus be achieved in a rational way, that is, the objective function should satisfy certain rational rules or 'consensus axioms'. In 1951 Arrow [1,2] initiated this approach for consensus functions by showing that certain sets of axioms could not be satisfied.

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2011.02.001

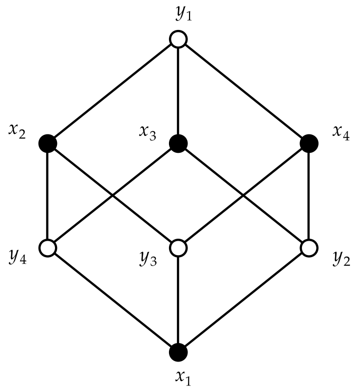

Fig. 1. The 3-cube and a balanced profile.

Three location functions have been studied axiomatically: the center function, the median function, and the mean function. For the center function [11,15] and the mean [4,19,8], characterizations have been obtained only on trees and tree networks. Characterizations beyond trees seem to be very difficult for these functions.

The median function is more promising. This function satisfies three simple and basic axioms: (A) Anonymity, the clients are anonymous; (B) Betweenness, any location strictly between two clients minimizes the sum of the distances to these two clients; and (C) Consistency, if two different sets of clients both prefer location x, then the union of all these clients also prefer location x. On most graphs and networks these axioms are not sufficient to characterize the median function. Hence the question arises: "On which graphs (networks) is the median function characterized by these three basic axioms?"

Vohra [19] obtained a characterization of the median function on tree networks using three simple axioms, which can be easily rephrased as (A)–(C). McMorris et al. [9] discussed the discrete case. They characterized the median function by means of (A)–(C) on an important class of graphs that generalizes trees, namely 'cube-free median' graphs. To extend this characterization to arbitrary 'median' graphs, the same authors [9] required a fourth less intuitively appealing 'convexity' axiom. In [6] another axiom, $\frac{1}{2}$-Condorcet, was introduced that did the same trick. For a survey of the results obtained so far see [10].

A median graph is a graph in which any three vertices have a unique median vertex. Besides trees, examples are the grid graphs and the hypercubes. Many applications of median graphs have been found in such diverse fields as biology, chemistry, literary history, social choice, economics, and location theory, see e.g. [13,6,7,14]. There is a rich structure theory for median graphs. Loosely speaking, a median graph can be obtained from a set of hypercubes by gluing these together along subcubes in a tree-like fashion. Note that a tree can be obtained by gluing together 1-dimensional cubes (edges) along 0-dimensional cubes (vertices) such that no cycle arises. A cube-free median graph does not contain the 3-cube Q_3, so only edges and 4-cycles are used in this gluing process.

In [9] an example is given of a set of four clients on the 3-cube Q_3. We recreate this example in Fig. 1 above, where the black vertices represent the four clients. In fact this provided a bottleneck: the proofs for the cube-free case did not work for this example, see [9]. Therefore a fourth axiom, Convexity, was introduced by McMorris et al. to make things work for arbitrary median graphs. Implicitly it was suggested that this example could also serve as counter-example for the arbitrary case. But basically it was an open problem: do the three axioms (A)–(C) suffice on the 3-cube to characterize the median function or is this fourth axiom really necessary?

In this note we settle this open problem. To our surprise, it turns out that the three basic axioms (A)–(C) are sufficient to characterize the median function on all hypercubes. The case for general median graphs remains open.

Recently a nice paper [3] appeared on the *remoteness function*: for a set of clients and a specific location the remoteness is the sum of the distances of this location to the clients. The median function minimizes remoteness, the *antimedian function* maximizes this value. This paper also explores this function on the hypercube. So there are similar ideas there. But the problems considered are different: in [3] metric properties of these functions are studied with emphasis on the antimedian function, here we seek axiomatic characterizations.

2. Preliminaries

Let $G = (V, E)$ be a graph. For any $u, v \in V$, we denote the geodesic distance between u and v by $d(u, v)$. The interval between v and u in G is the set

$$I_G(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\},$$

in other words all vertices 'between' u and v. When no confusion arises, we write $I(u, v)$.

A profile π of length k is a sequence $\pi = (x_1, x_2, \ldots, x_k)$ of vertices of V with repetitions allowed. A profile represents the location of the clients, where more than one client can be located at the same vertex. When π is a profile of finite length,