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Enumeration and asymptotics of restricted compositions having the same number of parts

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Dedicated to the memory of Philippe Flajolet.

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ABSTRACT

We study pairs and m-tuples of compositions of a positive integer n with parts restricted to a subset $\mathcal{P}$ of positive integers. We obtain some exact enumeration results for the number of tuples of such compositions having the same number of parts. Under the uniform probability model, we obtain the asymptotics for the probability that two or, more generally, m randomly and independently chosen compositions of n have the same number of parts. For a large class of compositions, we show how a nice interplay between complex analysis and probability theory allows to get full asymptotics for this probability. Our results extend an earlier work of Bóna and Knopfmacher. While we restrict our attention to compositions, our approach is also of interest for tuples of other combinatorial structures having the same number of parts.

1. Introduction

In this note, we study tuples of compositions of positive integers having the same number of parts, and the asymptotics of related generating functions satisfying some differential equations. Let us recall that a composition of a positive integer n is any k-tuple $(\kappa_1, \dots, \kappa_k)$, $k \geq 1$, of positive integers that sum up to n. The κ_j's are called the parts (or summands) of a composition. It is elementary and well-known (see, e.g. [1]) that there are $\binom{n-1}{k-1}$ compositions of n with k parts, and thus there are 2^{n-1} compositions of n. By restricted compositions we mean compositions whose parts are confined to be in a fixed subset $\mathcal{P}$ of $\mathbb{N}$. The main motivation for this work is a recent paper [6] in which the authors studied pairs of compositions with the same number of parts. Our extension of this work is directly connected to the question of obtaining the asymptotics of coefficients of functions satisfying a linear differential equation which, despite the deep work by Fabry, Frobenius, Fuchs, Picard and other analysts more than one century ago, remains open and is conjectured to be undecidable. We present here a new way to use probability theory *in addition to* complex analysis in order to solve this problem for a large class of functions. In their paper [6], Bóna and Knopfmacher studied the asymptotic probability that two randomly and independently chosen compositions of n have the same number of parts. Furthermore, relying on the generating function approach, for a few *specific* subsets $\mathcal{P}$ they addressed the same question for pairs of restricted compositions. In each of these

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cases this probability is asymptotic to $C/\sqrt{n}$ with C depending on $\mathcal{P}$. Our main aim here is to extend these results. First, we show that this asymptotics is universal. That is, we show that for an *arbitrary* subset $\mathcal{P}$ containing two relatively prime elements the probability that two independently chosen random compositions of n with parts restricted to $\mathcal{P}$ have the same number of parts is asymptotic to $C/\sqrt{n}$. The value of C depends, generally, on $\mathcal{P}$ and is explicit. (See our Theorem 5.1 and subsequent remarks, which include e.g. a correction of a constant appearing in [6].) Secondly, we consider the same question for $m > 2$ and we show that in this case the sought probability is asymptotic to $C/\sqrt{n^{m-1}}$ for an explicitly given constant C whose value depends on $\mathcal{P}$ and m only. (See our Theorem 5.3.)

Bóna and Knopfmacher's approach relied on complex analysis; the universality of using a more probabilistic technique was then noticed by Bóna and Flajolet [5], where certain types of random trees were studied. Our approach is in one sense a mixture of complex analysis (which gives the full asymptotics expansion, up to a multiplicative constant, and with the price of heavy computations), and probability theory (a local limit theorem which gives without any heavy computation the first asymptotic term, and therefore gives access to the multiplicative constant, but intrinsically no access to further asymptotic terms). Bóna and Flajolet obtained, in particular, a general statement indicating how local limit theorem can help in evaluating probabilities that two independently chosen random structures of the same size have the same number of components (this is their Lemma 6 in [5], which corresponds to our Lemma 5.2 for Gaussian density with a slightly different proof. Our Lemma 5.2 was obtained independently, but later). As we will see, these statements remain true if one considers more than two random structures.

In Section 2, we present our model. We proceed in Section 3 with some examples (and *en passant*, some nice questions in computer algebra) and argue on the intrinsic limitations of an approach relying only on complex analysis. This serves as a motivation for introducing the local limit law result in Section 4, which finds application in Section 5, thus solving the initial problem of the asymptotic evaluation of the probability that tuples of compositions have the same number of parts. We conclude with some perspectives in Section 6.

2. Generating functions for pairs of compositions having the same number of parts

Let us consider compositions with parts in a set $\mathcal{P}$ (a fixed subset of $\mathbb{N}$). To avoid trivial complications caused by the fact that there may be no compositions of a given n with all parts from $\mathcal{P}$, we assume that $\mathcal{P}$ has at least two elements that are relatively prime (except when explicitly stated otherwise).

We introduce the generating function of the parts $p(z) = \sum_{j\in\mathcal{P}} p_j z^j$, ($p_j$ is not necessarily 0 or 1, it can then be seen as the possible colors or the weight of part j). We thus assume that the p_j's are non-negative real numbers such that $\sum_{j\in\mathcal{P}} p_j > 1$. This last condition is to ensure supercriticality of our scheme (see Section 4 for more details). In the classical situation when p_j is 0 or 1, this condition holds automatically. Denote by

$$P(z,u) = \sum_{n\geq 0, k\geq 0} P_{n,k} u^k z^n = \frac{1}{1 - up(z)} \tag{1}$$

the bivariate generating function of compositions of n where k encodes its number of parts, and where the "size" of the composition is n.

With a slight abuse of notation, the corresponding univariate generating function is

$$P(z) = \sum_{n\geq 0} P_n z^n = \frac{1}{1 - p(z)}. \tag{2}$$

This terminology is classical. For example, here are all the compositions of 5 with 3 parts from the set $\mathcal{P} = \{1, 2, 3, 4, 10\}$: $5 = 1+1+3 = 1+3+1 = 3+1+1 = 1+2+2 = 2+1+2 = 2+2+1$. Accordingly, $P_{5,3} = 6$.

Let $X_n^{\mathcal{P}}$ be the random variable giving the number of parts in a random composition of n with parts belonging to $\mathcal{P}$. Random means that we consider the uniform distribution among all compositions of n with parts belonging to $\mathcal{P}$.

Given two subsets $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathbb{N}$, we consider the probability $\pi_n := \Pr(X_n^{\mathcal{P}_1} = X_n^{\mathcal{P}_2})$ that a random composition of n with parts in $\mathcal{P}_1$ has the same number of parts as a random composition with parts in $\mathcal{P}_2$. We assume throughout that, whenever two such compositions are chosen, they are chosen independently and from now on we will not be explicitly mentioning it. We then introduce the generating function $D(z)$ of the number of pairs of compositions (the first one with parts in $\mathcal{P}_1$, the second one with parts in $\mathcal{P}_2$) having the same size and the same number of parts. (D stands for "double" or "diagonal", as $D(z)$ can be obtained as a diagonal of multivariate function.)

That is, we consider all k-tuples of elements of $\mathcal{P}_1$ and all k-tuples of elements of $\mathcal{P}_2$ such that their sum is n. For a fixed n, let D_n be the total number of such configurations (i.e., we sum over all k).

In the next section, we deal with some interesting examples for which we get explicit formulas.

3. Some closed-form formulas

3.1. An example on tuples of domino tilings

Consider the classical combinatorial problem of tiling a $2\times n$ strip by dominoes. Any tiling is thus a sequence of either one horizontal domino or 2 vertical dominoes. The generating function of domino tilings is thus $P(z) = \mathrm{Seq}(z+z^2) = \frac{1}{1-z-z^2}$,