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On minimal vertex separators of dually chordal graphs: Properties and characterizations

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Many works related to dually chordal graphs, their cliques and neighborhoods were published by Brandstädt et al. (1998) [1] and Gutierrez (1996) [6]. We will undertake a similar study by considering minimal vertex separators and their properties instead. We find a necessary and sufficient condition for every minimal vertex separator to be contained in the closed neighborhood of a vertex and two major characterizations of dually chordal graphs are proved. The first states that a graph is dually chordal if and only if it possesses a spanning tree such that every minimal vertex separator induces a subtree. The second says that a graph is dually chordal if and only if the family of minimal vertex separators is Helly, its intersection graph is chordal and each of its members induces a connected subgraph. We also found adaptations for them, requiring just $O(|E(G)|)$ minimal vertex separators if they are conveniently chosen. We obtain at the end a proof of a known characterization of the class of hereditary dually chordal graphs that relies on the properties of minimal vertex separators.

1. Introduction

The class of chordal graphs has been widely investigated and several of its characteristics are very useful in the resolution of many problems. One example of them are phylogenetic trees [7,8].

Clique graphs of chordal graphs form a class considered in many senses as dual to chordal graphs, hence the name dually chordal graphs. Several studies about them were done, and as a result, many characterizations of dually chordal graphs were discovered, mainly involving cliques and neighborhoods. However, not much has been revealed about their minimal vertex separators. For that reason, one of the purposes of this paper is to study minimal vertex separators of dually chordal graphs to determine if the properties known about cliques and neighborhoods have their counterparts dealing with minimal vertex separators.

After reviewing some terminology and previous results in Sections 2 and 3, we describe in Section 4 all the results we could prove about minimal vertex separators of dually chordal graphs.

In 4.1, we show the first results of the approach described above. To the known fact that dually chordal graphs are endowed with spanning trees such that any clique or neighborhood induces a subtree, we add that the same is true for minimal vertex separators.

In 4.2, we study minimal vertex separators contained in neighborhoods. We see that many of them could be found with the help of the trees mentioned in the previous paragraph and we discover, among other results, that every minimal vertex separator of a dually chordal graph is contained in the neighborhood of a vertex if and only if every chordless cycle of length greater than or equal to four is contained in the neighborhood of a vertex.

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In 4.3, we can see how the results of 4.1 lead to new characterizations of dually chordal graphs. We prove that a graph is dually chordal if and only if there is a spanning tree of the graph such that any minimal vertex separator induces a subtree. Another necessary and sufficient condition is that every minimal vertex separator induces a connected subgraph and that all the minimal vertex separators form a Helly family whose intersection graph is chordal. We also look for weaker conditions and we find that these characterizations not always require considering all the minimal vertex separators, but a subfamily whose number of members is of the order of the number of edges of the graph.

Finally, in 4.4, we show how the characterizations appearing in 4.3 can be used to find the family of minimal forbidden induced subgraphs for the class of hereditary dually chordal graphs. This family was already known, but minimal vertex separators had never been used as a tool to find it.

2. Some graph terminology

This paper deals just with finite simple (without loops or multiple edges) graphs. For a graph G, $V(G)$ denotes the set of its vertices and $E(G)$ that of its edges. A *complete* is a subset of pairwise adjacent vertices of $V(G)$. A *clique* is a maximal complete and the family of cliques of G will be denoted by $C(G)$. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G.

Given two vertices v and w of G, the *distance* between v and w, or $d(v, w)$, is the length of any shortest path connecting v and w in G. The *open neighborhood* of v, or $N(v)$, is the set of all the vertices adjacent to v. The *closed neighborhood* of v, or $N[v]$, is defined by the equality $N[v] = N(v) \cup \{v\}$. The *disk* centered at v with radius k is the set $N^k[v] := \{w \in V(G),\ d(v, w) \leq k\}$.

Given two vertices u and v in the same connected component of G, a *uv-separator* is a set $S \subseteq V(G)$ such that u and v are in different connected components of $G - S := G[V(G) - S]$. It is *minimal* if no proper subset of S has the same property. We will just say *minimal vertex separator* to refer to a minimal set separating a pair of nonadjacent vertices. The family of minimal vertex separators of G is denoted by $\mathcal{S}(G)$.

Let G be a connected graph and let T be a spanning tree of G; for all $v, w \in V(G)$, $T[v, w]$ will denote the path in T from v to w or the vertices of this path, depending on the context. In the latter case, it is used to define $T(v, w)$ as the set $T[v, w] - \{v, w\}$.

Let $\mathcal{F}$ be a family of nonempty sets. $\mathcal{F}$ is *Helly* if the intersection of all the members of any subfamily of pairwise intersecting sets is not empty. If $C(G)$ is a Helly family, we say that G is a *clique-Helly graph*. The *intersection graph* of $\mathcal{F}$, $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ of G is the intersection graph of $C(G)$.

3. Basic notions and properties

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. *Chordal* graphs are those without chordless cycles of length at least four.

A vertex w is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. A linear ordering $v_1, \ldots, v_n$ of the vertices of G is a *maximum neighborhood ordering* of G if, for $i = 1, \ldots, n$, v_i has a maximum neighbor in $G[\{v_i, \ldots, v_n\}]$. *Dually chordal graphs* can be defined as those possessing a maximum neighborhood ordering. However, they were studied independently and simultaneously with different definitions and names, such as *HT-graphs*, *tree-clique graphs* and *expanded trees* [6,10]. Later, it became clear that the term *dually chordal* was the most fitting denomination for them [1].

The characterizations of dually chordal graphs are many. In fact, given a connected graph G, the following are equivalent [1]:

1. *G is dually chordal.*
2. *There is a spanning tree T of G such that any clique of G induces a subtree in T.*
3. *There is a spanning tree T of G such that any closed neighborhood of G induces a subtree in T.*
4. *There is a spanning tree T of G with any disk inducing a subtree in T.*
5. *G is clique-Helly and $K(G)$ is chordal.*

It is even true that any spanning tree fulfilling 2., 3. or 4. automatically fulfills the other two. Such a tree will be said to be *compatible* with G.

The characterization of a connected dually chordal graphs given by 3. can be rephrased as follows:

Theorem 1 ([6]). *A connected graph G is dually chordal if and only if there is a spanning tree T of G such that, for all $x, y, z \in V(G)$, $xy \in E(G)$ and $z \in T(x, y)$ implies that $xz \in E(G)$ and $yz \in E(G)$.*

4. On minimal vertex separators of dually chordal graphs

From now on, all the graphs considered will be assumed to be connected for a better handling of the proofs of the properties we are going to discuss. It is not difficult to extend them to disconnected graphs by applying the proofs to each connected component.