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On edge-sets of bicliques in graphs

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ABSTRACT

A *biclique* is a maximal induced complete bipartite subgraph of a graph. We investigate the intersection structure of edge-sets of bicliques in a graph. Specifically, we study the associated *edge-biclique hypergraph* whose hyperedges are precisely the edge-sets of all bicliques. We characterize graphs whose edge-biclique hypergraph is *conformal* (i.e., it is the clique hypergraph of its 2-section) by means of a single forbidden induced obstruction, the triangular prism. Using this result, we characterize graphs whose edge-biclique hypergraph is *Helly* and provide a polynomial time recognition algorithm. We further study a hereditary version of this property and show that it also admits polynomial time recognition, and, in fact, is characterized by a finite set of forbidden induced subgraphs. We conclude by describing some interesting properties of the 2-section graph of the edge-biclique hypergraph.

1. Introduction

The intersection graph of a collection of sets is defined as follows. The vertices correspond to the sets, and two vertices are adjacent just if the corresponding sets intersect. Intersection graphs are a central theme in algorithmic graph theory because they naturally occur in many applications. Moreover, they often exhibit elegant structure which allows efficient solution of many algorithmic problems. Of course, to obtain a meaningful notion, one has to restrict the type of sets in the collection. In fact, [19], every graph can be obtained as the intersection graph of some collection of sets. By considering intersections of intervals of the real line, subtrees of a tree, or arcs on a circle, one obtains interval, chordal, or circular-arc graphs, respectively. For these classes, a maximum clique or a maximum independent set can be found in polynomial time [9]. We note that one can alternatively define an interval graph as an intersection graph of connected subgraphs of a path; similarly intersection graphs of connected subgraphs of a tree produce chordal graphs, and intersection graphs of connected subgraphs of a cycle produce circular-arc graphs. More generally, one can consider intersections of particular subgraphs of arbitrary graphs. This naturally leads to intersections of edges, cliques, or bicliques of graphs which correspond to line graphs, clique graphs, and biclique graphs, respectively.

We focus on edge intersections of subgraphs. The edge intersection graph of a collection of subgraphs is defined in the obvious way, as the intersection graph of their edge-sets. In hypergraph terminology, this can be defined as the line graph of the hypergraph whose hyperedges are the edge-sets of the subgraphs. We say that subgraphs are edge intersecting if they share at least one edge of the graph. For instance, the EPT graphs from [10] are exactly the edge intersection graphs of paths in trees. For another example, consider the double stars of a graph G, i.e., the subgraphs formed by the sets of edges

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incident to two adjacent vertices. The edge intersection graph of double stars of G is easily seen to be precisely the square of the line graph of G. In contrast, if we consider the stars of G, i.e., sets of edges incident with individual vertices, then the edge intersection graph of the stars of G is the graph G itself [19].

In this context, one can study edge intersections of particular subgraphs by turning the problem into a question about vertex intersections of cliques of an associated auxiliary graph. In this auxiliary graph, vertices correspond to edges of the original graph G, and two vertices are adjacent just if the corresponding edges belong to one of the particular subgraphs considered. In the language of hypergraphs, this graph is defined as the two-section of the hypergraph of the edge-sets of the subgraphs. For instance, in line graphs vertices are adjacent if and only if the corresponding edges belong to the same star of G. A similar construction produces the so-called edge-clique graphs from [8] (see also [4–7,17,18]). Naturally, every occurrence of the particular subgraph in G corresponds to a clique in such auxiliary graph, and although the converse is generally false, one may obtain useful information by studying the cliques of the auxiliary graph.

Next, we turn our attention to the Helly property. A collection of sets is said to have the Helly property if for every subcollection of pairwise intersecting sets there exists an element that appears in each set of the subcollection. For instance, any collection of subtrees of a tree has the Helly property. On the other hand, arcs of a circle or cliques of a graph do not necessarily have the Helly property. Note that it is, in fact, the Helly property that allows us to efficiently find a maximum clique in a chordal graph or in a circular-arc graph (where the Helly property is "almost" satisfied [9]). By comparison, finding a maximum clique appears to be hard in clique graphs (intersection graphs of cliques). For a similar reason, recognizing chordal graphs and circular arc graphs is possible in polynomial time [9], whereas it is hard for clique graphs [1].

Alternatively, one can impose the Helly property on intersections, and then study the resulting class of graphs. For instance, cliques of a graph do not necessarily satisfy the Helly property, but if we only consider graphs in which they do, we obtain the class of clique-Helly graphs studied in [16]. In the same way, one can study the classes of neighbourhood-Helly, disc-Helly, biclique-Helly graphs [11], and also their hereditary counterparts [12,15].

In this paper, we investigate the intersections of edge-sets of bicliques. With each graph G we associate the *edge-biclique hypergraph*, denoted by $\mathcal{EB}(G)$, defined as follows. The vertices of $\mathcal{EB}(G)$ are the edges of G, and the hyperedges of $\mathcal{EB}(G)$ are the edge-sets of the bicliques of G. We remark that while for cliques the usual vertex intersection graphs (i.e., clique graphs and hypergraphs) are the most natural construct, for bicliques both the vertex and the edge intersection graphs are natural, and have interesting structure. (See [13] for a characterization of vertex intersection graphs of bicliques.)

The paper is structured as follows. First, in Section 2 we observe some basic properties of the two-section graph of the edge-biclique hypergraph $\mathcal{EB}(G)$. This will allow to prove that $\mathcal{EB}(G)$ is *conformal* (it is the hypergraph of cliques of its two-section) if and only if G contains no induced triangular prism. Next, in Section 3 we discuss the Helly property and prove that $\mathcal{EB}(G)$ is Helly if and only if the clique hypergraph of the two-section of $\mathcal{EB}(G)$ is Helly. This will imply polynomial time testing for the Helly property on $\mathcal{EB}(G)$. In Section 4 we look at a hereditary version of this property by studying graphs G such that for every induced subgraph H of G, the hypergraph $\mathcal{EB}(H)$ is Helly. We show that the class of such graphs admits a finite forbidden induced subgraph characterization. This will also yield a polynomial time recognition algorithm for the class. In Section 5, we conclude the paper by further discussing properties of the two-section graph of $\mathcal{EB}(G)$. In particular, we compare it to the line graph of G, point out some small graphs that are not two-sections of edge-biclique hypergraphs, and characterize graphs whose every induced subgraph is the two-section of some edge-biclique hypergraph.

2. Notation and basic definitions

A *graph* $G = (V, E)$ consists of a vertex set V and a set E of edges (unordered pairs from V). A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a vertex set V and a set $\mathcal{E} \subseteq 2^V$ of hyperedges (subsets of V). For a set X of vertices of a graph G, we denote by $G[X]$ the subgraph of G induced by X. A set X is a *clique* of G if $G[X]$ is a complete graph and X is (inclusion-wise) maximal with this property. A set X is a *biclique* of G if $G[X]$ is a complete bipartite graph and X is (inclusion-wise) maximal with this property.

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a subset $\mathcal{E}' \subseteq \mathcal{E}$, we say that $\mathcal{H}' = (V, \mathcal{E}')$ is a *partial hypergraph* of $\mathcal{H}$. A *subhypergraph* of $\mathcal{H}$ *induced* by a set $A \subseteq V$ is the hypergraph $\mathcal{H}[A] = (A, \{X \cap A \mid X \in \mathcal{E}\} \setminus \{\emptyset\})$.

To make the presentation clearer, we shall use capital letters $G, H, \ldots$ to denote graphs and calligraphic letters $\mathcal{G}, \mathcal{H}, \ldots$ to denote hypergraphs. Similar convention shall be used for graph and hypergraph operations. In particular, the following operations shall be used throughout the paper.

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. The *dual hypergraph* of $\mathcal{H}$, denoted by $\mathcal{H}^*$, is the hypergraph whose vertex set is $\mathcal{E}$ and whose hyperedges are $\{\mathcal{X}_v \mid v \in V\}$ where $\mathcal{X}_v = \{X \mid X \in \mathcal{E} \wedge X \ni v\}$. In other words, each $\mathcal{X}_v$ consists of all hyperedges of H that contain v. The 2-*section* of $\mathcal{H}$, denoted by $(\mathcal{H})_2$, is the graph with vertex set V where two vertices $u, v \in V$ are adjacent if and only if $u, v \in X$ for some $X \in \mathcal{E}$. The *line graph* of $\mathcal{H}$, denoted by $L(\mathcal{H})$, is the graph with vertex set $\mathcal{E}$ where $X, X' \in \mathcal{E}$ are adjacent if and only if $X \cap X' \neq \emptyset$. Note that $L(\mathcal{H})$ is the 2-section of the dual hypergraph of $\mathcal{H}$.

Let $G = (V, E)$ be a graph. The *line graph* of G, denoted by $L(G)$, is the graph with vertex set E where two edges of E are adjacent if and only if they share an endpoint in G. The *clique hypergraph* of G, denoted by $\mathcal{K}(G)$, is the hypergraph whose vertex set is V and whose hyperedges are the cliques of G. The *clique graph* of G, denoted by $K(G)$, is the graph whose vertices are the cliques of G where two cliques are adjacent if and only if they have a vertex in common. In other words, $K(G)$ is the line graph of the clique hypergraph $\mathcal{K}(G)$. The *edge-biclique hypergraph* of G, denoted by $\mathcal{EB}(G)$, is the hypergraph with vertex set is E whose hyperedges are the edge-sets of the bicliques of G. The *biclique line graph* of G, denoted by L_G, is the