Discrete Applied Mathematics 160 (2012) 2778–2786

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Wirelength of hypercubes into certain trees

Indra Rajasingh [a], Paul Manuel [b], Bharati Rajan [a], M. Arockiaraj [a,*]

[a] Department of Mathematics, Loyola College, Chennai 600 034, India
[b] Department of Information Science, Kuwait University, Safat, Kuwait

ARTICLE INFO

Article history:
Received 28 April 2010
Received in revised form 1 November 2011
Accepted 8 December 2011
Available online 31 December 2011

Keywords:
Embedding
Congestion
Wirelength
Edge isoperimetric problem
Hypercubes

ABSTRACT

A lot of research has been devoted to finding efficient embedding of trees into hypercubes. On the other hand, in this paper, we consider the problem of embedding hypercubes into k-rooted complete binary trees, k-rooted sibling trees, binomial trees and certain classes of caterpillars to minimize the wirelength.

1. Introduction

An important feature of an interconnection network is its ability to efficiently simulate programs written for other architecture. Such a simulation problem can be mathematically formulated as graph embedding. An embedding of a guest graph G into a host graph H is defined by an injective mapping $f: V(G) \to V(H)$ together with a mapping P_f which assigns to each edge (u, v) of G a path $P_f((u, v))$ between $f(u)$ and $f(v)$ in H [9,29,33]. Some of the parameters used to analyze the efficiency of an embedding are dilation, expansion, edge congestion and wirelength.

If $e = (u, v) \in E(G)$, then the length of $P_f(e)$ in H is called the dilation of the edge e. The maximal dilation over all edges of G is called the dilation of the embedding f. The expansion of an embedding f is the ratio of the number of vertices of H to the number of vertices of G. In this paper, we consider embeddings with expansion one.

The *edge congestion*of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(e)$ denote the number of edges (u, v) of G such that e is in the path $P_f((u, v))$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(e) = \left|\left\{(u, v) \in E(G): e \in P_f((u, v))\right\}\right|.$$

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f: V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [5].

The wirelength [21,25] of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v)\in E(G)} \left|P_f((u, v))\right| = \sum_{e\in E(H)} EC_f(e).$$

* Correspondence to: Department of Mathematics, Loyola College, Nungambakkam, Chennai 600 034, India. Fax: +91 9840298901.
E-mail address: marockiaraj@gmail.com (M. Arockiaraj).

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doi:10.1016/j.dam.2011.12.007

The *wirelength* of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. The *wirelength problem* of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$. Since our goal to construct embeddings of minimum wirelength, we will take P_f to be a mapping that assigns to each edge (u, v) of G a shortest path between vertices $f(u)$ and $f(v)$ in H.

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [24,33]. Embedding problems have been considered for binary trees into paths [24], binary trees into hypercubes [3,7,13,14,17,20,22,26], binomial trees into hypercubes [31,32], generalized ladders into hypercubes [10], binary trees into grids [27], hypercubes into cycles [15,19], generalized wheels into arbitrary trees [28], and hypercubes into grids [25].

Even though there are numerous results and discussions on the embedding problem, most of them deal with only approximate results and the estimation of lower bounds [4,15]. The embeddings discussed in this paper produce exact wirelength.

2. Isoperimetric problem

The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [6], which is NP-complete [18].

Version 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $\theta_G(m) = |\theta_G(A)|$.

It is interesting to note that $\theta_G(\lfloor |V|/2 \rfloor)$ solves bisection width of G [18].

Version 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $I_G(m) = |I_G(A)|$.

We call such a set A optimal. Clearly, if a subset of vertices is optimal with respect to Version 1, then its complement is also an optimal set. However, it is not true for Version 2 in general, although this is indeed the case if the graph is regular [6]. In the literature, Version 2 is defined as the maximum subgraph problem.

The hypercube is one of the most popular versatile and efficient topological structures of interconnection networks. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machine based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [30].

Definition 1 (*[33]*)**.** For $n \geq 1$, let Q_n denote the graph of n-dimensional hypercube. The vertex set of Q_n is formed by the collection of all n-dimensional binary representations. Two vertices $x, y \in V(Q_n)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit.

Equivalently if $|V(Q_n)| = 2^n$ then the vertices of Q_n can also be identified with integers $0, 1, \ldots, 2^n - 1$ so that if a pair of vertices i and j are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

Definition 2 (*[23]*)**.** An incomplete hypercube on i vertices of Q_n is the subcube induced by $\{0, 1, \ldots, i - 1\}$ and is denoted by L_i, $1 \leq i \leq 2^n$.

Theorem 1 (*[8,11,21]*)**.** *Let Q_n be an n-dimensional hypercube. For $1 \leq i \leq 2^n$, L_i is an optimal set.* □

Lemma 1 (*[2,25]*)**.** *Let Q_n be an n-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $n \geq t_1 > t_2 > \cdots > t_l \geq 0$. Then $|E(Q_n[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \cdots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l-1)2^{t_l}]$.* □

In this paper we solve the wirelength problem of hypercubes into k-rooted complete binary trees, k-rooted sibling trees, binomial trees and certain classes of caterpillars. We begin with the following notation.

Notation. For any set S of edges of H, $EC_f(S) = \sum_{e\in S} EC_f(e)$.

Lemma 2 (*Congestion Lemma [25]*)**.** *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*